The Arithmetic Teacher

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ETHEL GUNDERSON

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JOURNAL OF THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

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THE ARITHMETIC TEACHER

Volume V Number 5

November 1958

Fractions—Seven-Year-Olds Use Them

MRS. ETHEL GUNDERSON

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THIS STUDY IS A FOLLOW-UP of an earlier one: "Fraction Concepts Held by Young Children" which was reported in the October 1957 issue of THE ARITHMETIC TEACHER. It is an attempt to find out whether seven-year-olds, after a brief introduction to fractions, are able to extend the ideas which are developed in class and incorporate them into their own experiences.

An introductory lesson such as that described in the earlier study was taught to a group of 23 second graders in the Willets Road School, East Williston, Long Island, N. Y. in February 1958.

Following this introductory lesson in which the children cut "pies" (paper circles) in halves, fourths, eighths, thirds, and sixths, the teacher gave several oral problems which the children solved using paper circles which they folded or cut into the various fractional parts.

Original Problems (oral)

After the children had solved the problems given by the teacher, they were asked to make up problems of their own and tell them to the other children in the group who would work them out and give the answer. Each child was provided with several paper circles (all the same size) to use in solving the problems. If the pie was first cut in fourths, they would do this

(everyone cut his paper pies). Now if a fourth was cut in half they all did this so they would see that the pie had 5 pieces and the name of the piece as "one fourth" and "one eighth"

was clear to them. When there was the problem of cutting pie in eight pieces they did this,

and as the pieces were "eaten" (in the problem) they would take those pieces away so they could see the eighths that were remaining.

Some of these problems with solutions and discussions are given below:

GARY: I had a whole pie and I invited Jay for dinner. My mother was there and she wanted pie so we cut the pie in *thirds*. Jay said, "That piece is too big; cut it in *half*." So I cut the third piece in *half*. How much of the pie did Jay eat?

Answer: (Steve) *One sixth* of a pie, or you could say *one half of a third*.

GARY: How much did I eat?

Answer: You ate *one third* and your mother also ate *one third*.

Teacher: Was there any pie left?

Answer: Yes, of course, there is *one sixth* of a pie left; the part Jay didn't want.

WENDY A.: I invited the twins over to play with me. Mother said we could have some pie. So I cut the *whole* pie in three pieces. Josie said, "I can't eat all that." Randi said that she could eat all her pie and the *half* that Josie could not eat. So I gave Josie's to Randi. Now I want to know how much pie Randi ate?

Answer: (Carol) I know; Randi ate *one half* pie.

Teacher: How can you know that, Carol?

Answer: Well, she had her own *third* of a pie and Josie's *half of a third* which is *a sixth*, and when I put *one third* of a pie and *one sixth* of a pie together on my desk it is *a half* pie.

Teacher: How much did Josie eat?

Answer: She only ate *one sixth*—this piece is what she ate.

Teacher: What part did Wendy eat?

Answer: Oh, she ate *a third*.

ERIC: One day I was walking by a bakery and I saw a pie. I had enough money so I bought it. I brought it home to my mother. Then my father called and said he'd be home for dinner. So the pie was ready for dessert. My brother, me, my sister, my father, and my mother were there, so that's five people. But my mother cut the pie in *thirds*. Then she cut *one of the thirds in half*.

Teacher: What would you call the size of those pieces?

Answer: Well, you call that *one sixth*. There were *two sixths* and me and my sister ate that part of the pie. My mother ate *one third* of a pie; my father ate *one third* of a pie. My mother didn't give any pie to my little brother; he had something else.

Teacher: How much of the pie did your parents eat?

Answer: Oh, they ate *one third* and *one third* or *two thirds* of the pie.

WENDY B.: Last Saturday (make believe) was my birthday and one of the things we had was pie. I invited eight children but only seven came. We cut the pie in eight pieces but two children did not want pie. How many pieces were left?

Answer: Two pieces were left.

Teacher: What do we call those pieces?

Answer: They are *eighths*. *Two eighths* were left; *six eighths* were eaten. You know there are eight pieces in the *whole* pie.

Teacher: Which would you rather have, *two eighths* of a pie or *one fourth* of a pie?

Answer: Well, I'd take the *two eighths* because I'd rather have two small pieces than one big one, but it is really the same amount. You can see that when you put *two eighths* on top of *one fourth* it is the same.

JANET P.: One day we had apple pie and I invited two people over. So we had the apple pie for dessert. We cut the pie in *fourths*. You know there are four people in my whole family and two more people makes six. So we had these four pieces of pie and then my mother cut *one of the fourths* in half. Now how many pieces of pie do we have?

Answer: You have five pieces now.

JANET P.: I forgot to say my father doesn't eat pie because he doesn't want to get fat.

Teacher: So five people had pie?

JANET P.: Yes, me and my mother had the small pieces.

Teacher: What would you call the small pieces?

JANET P.: I call them *eighths*. Mom and me each had *an eighth* or *two eighths*.

Teacher: How much pie did the other people have?

Answer: They each had *one fourth* of a pie.

KEN: I came home from school and I asked my mother did we have anything for a snack. She said, "Yes, we have a pie." Then my friend called; he wanted to come over. So he came over. So we were six people altogether and my mother cut the pie in eight pieces. My friend did not want pie; my brother did not want pie. I want to know how many pieces were left?

Answer: *Four eighths* or *a half* pie was left.

Teacher: If Ken's brother had eaten a piece, how much pie would be left?

Answer: Well, then *three eighths* would be left; and if the other kid would have eaten a piece there would be *two eighths* left, which is, as you must all know by now, *one fourth*.

JIM: Once we had a pie and we cut it in three pieces. My brother had one piece, I had one piece, and one piece fell on the floor and the dog ate it. How much did the children eat?

Answer: They ate *two thirds* of the pie.

JANET B.: Last Sunday my grandma and grandpa came. My mother had a pie in the refrigerator for dessert. There are six in my family and with Grandma and Grandpa there were eight people. We cut the pie in *eighths*. My baby brother does not eat pie. I want to know how much pie did we eat?

MALLORY: Did seven people eat pie?
JANET B.: Yes.
Answer: You used *seven eighths* of the pie.
JANET B.: How much was left?
Answer: *One eighth* of the pie was left.
Teacher: If only Janet's family had been having Sunday dinner, how do you think they would have cut the pie?
ERIC: Well, there are six people in the family so I suppose they would cut it in sixths, but Janet said the baby doesn't eat pie so they'd have one piece left.
Teacher: What part of the pie would be left, Eric?
Answer: *One sixth* would be left.

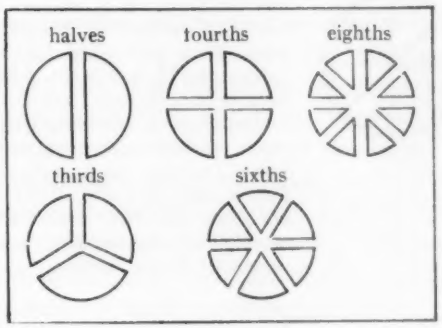
PETER: One morning my friend came over and we went with my mother shopping and she bought a pie. My whole family is four and this friend makes five. We cut the pie in *sixths* but my friend said he did not want pie, so my question is: How much pie did we eat and how much pie was left?
Answer: You ate *four sixths* of the pie and you have *two sixths* left.

MALLORY: One day my sister was going out so my mother said I could have a little party so I invited two friends and my mother served pie. It was a kinda small pie so she cut it in *fourths*. How much of the pie was served, and how much was left?
ARLENE: I don't get how many people were served?
MALLORY: Me and my two friends were served.
Answer: Oh, then *three fourths* were served and *one fourth* was left.

STEVIE: I was walking to the Super Market. My mother said, "Buy these things and you may keep the left-over money." So I had 59 cents left over. I was walking around to see what I could buy and I saw a pie that cost 59¢, so I decided to buy it for a surprise for my mother. So I came home and I said, "I bought you a cherry pie." She said, "I don't believe you"; but there was the pie. We cut it in *sixths*. My brother took a piece; I took a piece; my mother took a piece; my father took a piece. How much pie did we eat?
Answer: *Four sixths*, or you could say *two thirds* because it means the same.
STEVIE: How much pie was left?
Answer: *Two sixths* or you could say *one third* was left.

Original Problems (written)

After the introductory lesson and the solving of oral problems, the children spent much of their free time playing with the fractional parts of flannel at the flannel board. Also available were all the paper "pies" they wanted. Interest was very high, particularly among the more mature pupils. One child was overheard to say, "This is the greatest arithmetic." Also in the classroom was a large chart showing halves, fourths, eighths, thirds, and sixths, so the



spelling of the names of these fractional parts was there for them to see and consult in the writing of their own arithmetic problems.

These children had been writing original stories, poems, and letters since Christmas time. To express themselves in writing was almost a daily occurrence. They had previously written original arithmetic problems using addition; now they went to work writing, illustrating, and answering problems about fractions. Time is given for each pupil to read his problem to the class and call on someone to tell the answer.

Following are some of the problems written by the children. All of these problems had the correct answers and some of them were accompanied by illustrations. Only a minimum of editing has been done on them, e.g. inserting a period or comma to clarify the problem. Quotation marks are those put in by the child.

(*Problems continued on next page*)

JOAN F.:

One day I was walking home from school, and I passed the bakery. Then I stop and said 'it's my mother's birthday today! so I will surprise her and bring home a pie saying with whipped cream Happy birthday mother." So I walked in and said, "Miss, I want a birthday pie." So she gave me my pie. Then I got home; I shouted "surprise!" and we cut the pie in fourths. I had 1 fourth and my mother had 1 fourth. Then my father and sister said we'll have the same. How much was left? None was left.

MARK:

Once there was a little boy, His allowance was two dollars. One day the boy saw a bakery. He asked his mother for his allowance. Then he walked into the bakery and he saw a pie. M-M-m-m he said. So he bought it. And he had a party; two of his friends came. And his mother and father came. And they cut it in eighths. Five people came, and all of them ate some pie. How many were eaten? Five eighths. How many were left? Three eighths.

DAVID:

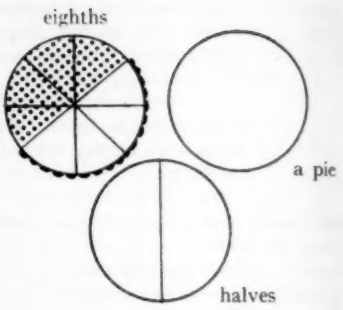
I was coming from school and I bought a pie and I brought it home. I was having Peter, Kenny, and Mark over to eat some pie. My mother had some too and so did I. So I cut it in sixths. Mark did not want any. How pieces of pie were eaten? 4 sixths. How many were left? 2 sixths.

ARLENE:

My friend Jill came over and I wanted something to eat. So I took a pie out of the freezer. Jill wanted a third of a pie. I wanted a third of a pie too. It was good. How much did we eat? Two thirds. And how much did we have left? One third.

MALLORY:

I had a pie and I cut it into eighths. My friend called up. She wanted to come over to have some pie. So my friend did come over and had some pie. I gave my friend 3 eighths and she finished all of it. And my friend wanted some more pie and I gave one more eighth. How much did I have for me? I have a half of pie.



DONALD:

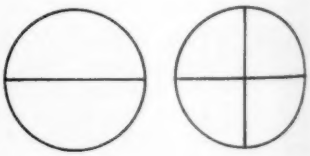
Once I came from school. I passed a bakery. And inside, I saw a pie for $1.25, so I got it. So then I went home. My mother cut the pie in sixths. I said "mother what are you doing?" My mother said "we are having company tonight. There will be five people including us." So they each had one piece. How many pieces are left? 1 piece. How many are eaten? 5 pieces.

PAUL:

If I had a pie cut in thirds and I gave you 2 thirds, how much is left? 1 third.

GARY:

One day I came home from school. My mother wasn't home, so I went outside to play. That night I came home I saw a pie, it was cut in halves. There are four people in my family including me. How would I cut my pie to have enough for the whole family? Cut it in fourths.



CAROL:

One day I was coming home from school. I passed the shoe shop, and the bakery. I stopped at the bakery, and looked in the window, and what do you think I saw? A pie that cost 5 cents. I bought the pie and brang it home. My mother cut the pie in eighths. she gave one eighth to one sister, and one eighth to my other sister,

and one eighth to my father, and one eighth to me and to herself. How much was left? How much did we eat? 3 were left. We ate 5 eighths.

WENDY A.: We have some new neighbors on our block. We were invited to their house. They had a pie. They cut it into eighths. I had one. My father had one. My mother had one. My sister had one. The new neighbor had one and his wife had one. How many were left? 2 eighths. How many did we eat? 6 eighths.

STEVE: Once I was going to New York to eat lunch. We had pie for dessert. There were four people there. We cut the pie in fourths. But my brother didn't want any. How many didn't we eat? 1 fourth. How many did we eat? 3 fourths.

ERIC: One day I was walking by the bakery and I saw a pie. So I asked the man how much it was. He said it is a dollar. I had two dollars. So I bought it. And then I went home. And my mother said "do you want to invite five of your frinds over?" "Yes I do." So we cut it in sixths. But one of my friends did not like it. How much did we eat? How much did we have left? We ate five sixths of a pie. We had one sixth left.

KENNETH: One day I came home from school. My mother had pie for snack. My mother cut the pie in sixths. My brother wanted some pie and my mother wanted some pie and my father wanted some pie and I wanted some pie. We each had one sixth. How many sixths were left? Two sixths. How many were eaten? Four sixths.

PETER: One day I passed the bakery with my mother. We went in. I saw a blue-berry pie. I asked my mother if she buy it. She did. She cut it in thirds. I gave my mother 1 third. I had 2 thirds. Who had more, me or mother? I had more pie than mother.

Conclusions

All the children in the group solved correctly (with the use of manipulative materials) problems dealing with fractions, which were given by the teacher. The less mature pupils cut the fractional parts correctly for the more simple problems given by the teacher, but did not volunteer to make problems nor to answer the problems made by the other children. These will need more work similar to the introductory lesson under the teacher's guidance before they are able to work independently in making problems.

That so many of the pupils contributed original problems indicates that many seven-year-olds are able to extend the fractional concepts and ideas which are developed in class when manipulative materials are used. No algorisms should be used at this stage as this would only cause confusion and bewilderment. The fraction symbols: $\frac{1}{2}$, $\frac{3}{4}$, etc. are far too advanced for these young children, but they can understand the fraction when expressed as 1 half, 3 fourths, etc.

The problems solved by these seven-year-olds involve: addition of fractions, subtraction of fractions, and fractional equivalents. Partitive division is called for in dividing a pie into the fractional parts needed, e.g. cut the pie in half, cut one of the fourths in half, cut one of the thirds in half. The computation involved was not too difficult for these children because they worked with manipulative materials: wholes (all the same size) which they could cut into the desired fractional parts. These fractional parts could be grouped together, taken away, or superimposed upon one another. These same computations expressed in algorisms would be difficult enough for a fourth or fifth grader as they are, such as: $1-\frac{2}{3}=\ ?$;

$\frac{1}{3}+\frac{1}{6}=\frac{}{2}$; $\frac{4}{6}=\frac{}{3}$; $\frac{2}{8}=\frac{}{4}$; $\frac{1}{6}=\frac{1}{2}$ of ?

The conclusions reached in this study are similar to those reached in the earlier study

referred to, and emphasize again the need for a long acquaintance period between the child's first introduction to fractions and the time he is expected to work fractions using algorisms. This acquaintance period should provide for planned, systematic work with manipulative and semi-concrete materials. Other shapes, e.g. squares, rectangles, as well as circles may be used. Fractions of a group as well as fractions of a whole may be studied later, such as taking one half of 4 counters, one third of 6 marbles and so on. Many opportunities should be given for children to make and solve original problems dealing with fractions.

These children have been encouraged to use the fractional terms in their daily conversation. For instance, during snack period which occurs every morning at 10:00 o'clock there is much trading and sharing. Frequently they have been overheard using fractions as: "Give me half of your cookie and I will give you half of my potato chips." "Give me one fourth of your cup cake and I will give you a half piece of gum." "Paul, divide your jelly beans in thirds." "This apple is worth more than half of your candy bar." "Fill the jar nearly to the top—make seven-eighths full." "I'll give you this tin foil that you can make a lot of things with if you'll give me just a third of your licorice." (The answer to this was, "No.")

Last but not least, the children thoroughly enjoyed these lessons. They evoked the most enthusiastic response to arithmetic observed during the entire year. To these children ARITHMETIC IS FUN.

EDITOR'S NOTE. Readers may wish to reread the article "Fraction Concepts Held by Young Children" which appeared in THE ARITHMETIC TEACHER, October, 1957. At that time the editor wondered how these particular pupils might retain and extend the knowledge acquired in 1957. Now Mrs. Gunderson has given us the answer. Apparently the type of teaching which features understanding through the use of visual-manipulative materials was not only retained but was extended by many of the pupils. The mode of learning is very important. Mrs. Gunderson points out that it is desirable to provide for a considerable span of time from the beginning of the development of concepts to the more final stage of working with abstract symbols.

It is easily apparent that children can and do think with fractions much as they do with whole numbers when the complication of the traditional symbolism is not present. Many a housewife would be better off if she could visualize fractional amounts and think with them as did these seven-year-old children. Too many adults become disturbed by trying to think symbolically only. These pupils should, in the next four years, extend their concepts and methods of work with fractions just as they will their appreciation of syntax.

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"Sets" and Elementary School Mathematics

NORBERT LERNER, *Cortland State Teachers College, N. Y.*
MAX A. SOBEL, *Montclair State Teachers College, N. J.*

IN RECENT YEARS mathematicians have been talking a good deal about "sets" and some people have been urging the reorganization of arithmetic instruction to include not only sets but also certain other concepts of modern mathematics. It has been urged that the introduction of sets will serve as a unifying concept for the fundamental ideas of arithmetic, algebra, and some forms of geometry and thus make it easier for a student to deal with mathematics as a unified field instead of a collection of several different areas of study.

So that elementary school teachers may understand better what the mathematicians are talking about and so that they may see some of the implications and possibilities for work in the elementary school this paper has set the following objectives.

1. To present in as readable a style as possible the basic concepts and definitions related to the concept of a set so that every teacher of elementary mathematics will be familiar with the language being proposed, and the ideas that this language conveys.

2. To indicate possible implications of sets for the teaching of mathematics in the elementary and junior high school.

3. To suggest further readings for those interested in a more detailed account of the subject.

Basic Definitions and Concepts

This section, dealing with the basic concepts and definitions of sets, is presented for the reader's background; it is not being suggested as proper subject matter for the elementary grades. On the other hand, many of the concepts to be described may be introduced at a very early stage on an intuitive basis as will be shown later.

By a set we shall mean a collection of elements. Thus we may speak of:

a. the set of students in the school
b. the set of girls with blond hair in the classroom
c. the set of even integers less than 20
d. the set of good ball players

In examining the preceding illustrations of sets we note that the last set differs from the others in that it is not a clearly defined set. That is, we are unable to list the elements of this set; each person's concept of "good" differs. Each of the other sets are, however, well defined; from the description the reader is able to list the elements of the set.

Next consider the set of odd integers. This is a well defined set yet differs from the others listed above in that its elements comprise an infinite collection of numbers. One can, however, list as many of the elements as desired: 1, 3, 5, 7, 9, · · · .

We can describe sets whose elements include every item under consideration. The set of human beings who breathe would obviously be an example of such a set. Another might be the set of even integers divisible by two; by definition this would include all of the even integers. On the other hand it is possible to name or describe a set which contains no elements, such as the set of odd integers divisible by two or the set of human beings who have visited the moon. Such a set is called a *null set* and frequently is denoted by the symbol "$\varnothing$."

To simplify discussion we introduce the following symbolism: we shall name a set with a capital letter and list its elements within a pair of braces. For example:

a. The set of names for the days of the week: $W = \{$Monday, Tuesday, Wednesday, Thursday, Friday, Saturday, Sunday$\}$.

b. The set of odd integers less than 8: $A = \{1, 3, 5, 7\}$.

c. $B = \{5, 6, 7, 8, 9\}$: B is the set of integers between 4 and 10, or alternatively B is the set composed of the elements 5, 6, 7, 8 and 9.

Two other basic definitions are necessary before we proceed.

By a *subset* of a set we shall mean a set composed of either part or all of the original set. For example, if $A = \{\text{apple, pear, orange}\}$, then we can list the following subsets of A: $\{\text{apple}\}$, $\{\text{pear}\}$, $\{\text{orange}\}$, $\{\text{apple, pear}\}$, $\{\text{apple, orange}\}$, $\{\text{pear, orange}\}$, $\{\text{apple, pear, orange}\}$. Again, let $R = \{1, 2, 3\}$. We can list the following subsets of R:

$$\{1\},\ \{2\},\ \{3\},\ \{1, 2\},\ \{1, 3\},\ \{2, 3\},\ \{1, 2, 3\}$$

Normally the null set, $\varnothing$, which consists of no elements, may be considered to be a subset of every set. Formally we define a subset as follows:

A set B is said to be a subset of a set A if every element of B is also an element of A.

Example: Let set $A = \{3, 6, 9, 12, 15\}$
Let set $B = \{3, 6, 9\}$

Then set B is a subset of set A because each element of B is a member of set A. The set $C = \{3, 12, 15\}$ is also a subset of A. Also the set $D = \{3, 6, 9, 12, 15\}$ is a subset of set A.

From the definition given above it follows that every set may be considered to be a subset of itself. When this possibility is to be ruled out we do so by introducing the concept of *proper subset.*

A set B is said to be a proper subset of a set A if every element of B is an element of A and there are elements of A which are not elements of B.

Informally, the concept of proper subset states that one set is part of a second set but does not exhaust all the members of the second set.

Example: Let set $A = \{3, 6, 9, 12, 15\}$
Let set $B = \{3, 6, 9\}$

Then set B is not only a subset of A, it is also a proper subset of A, since there are members of A, namely 12 and 15, which are not in B. The set $D = \{3, 6, 9, 12, 15\}$ is a subset of A but is *not* a proper subset because it includes every member of A.

Consider next the set A consisting of the brown-eyed students in a classroom, and B representing the set of students in that room who have visited Washington, D. C. *By the union of sets A and B we mean the collection of elements which are members of at least one of the sets.* We write this symbolically as $A \cup B$, read "A union B." The collection of elements in this union is the set of students who have either visited Washington or have brown eyes. It may be that some students have made the visit and also have brown eyes; this does not bother us. We do know, however, that every student in $A \cup B$ has at least one of these attributes.

By the intersection of sets A and B we shall mean the collection of elements which are in both A and B. We write this as $A \cap B$, read "A intersection B." Thus $A \cap B$ would be those brown-eyed students who have also visited Washington. Thus every member of the intersection has both of these characteristics.

As another illustration consider the following two sets:

$$A = \{1, 2, 3, 4, 5, 6, 7\} \qquad B = \{5, 6, 7, 8, 9\}$$

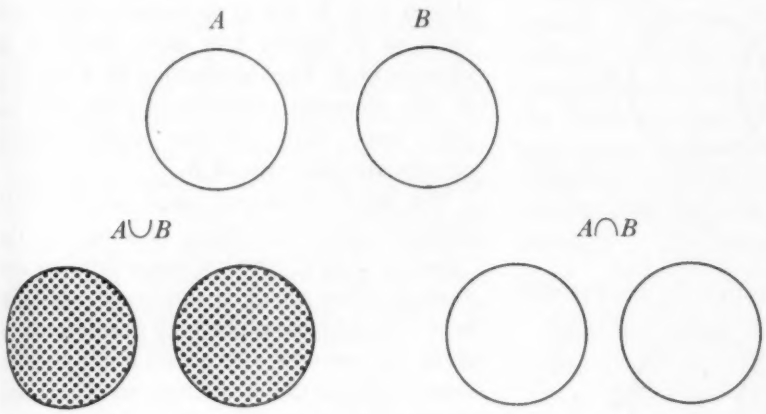
Then

$$A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

(each element of this union appears in at least one of the original sets), and $A \cap B = \{5, 6, 7\}$ (each element of this intersection appears in both of the original sets). Note that the intersection of A and B is a subset of A and is also a subset of B in that its elements are members of both of these sets. Also, both sets A and B are subsets of the union of A and B.

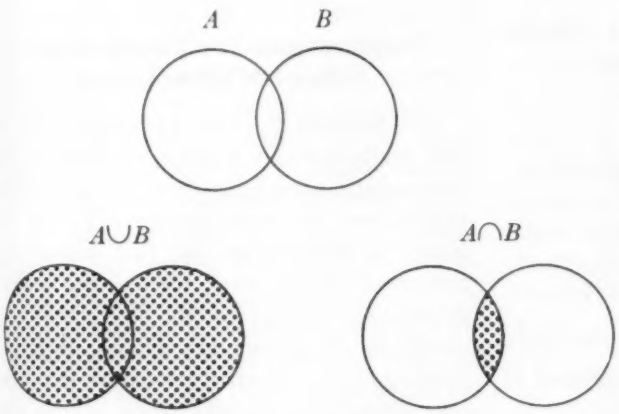
A visual interpretation is offered us by means of circles usually called "Venn" diagrams. We represent two sets as circles whose elements, in this case points, lie within the circle and consider several possibilities. In each case the union and intersection of the sets are indicated by the shaded areas.

1. Sets A and B have no points in common:



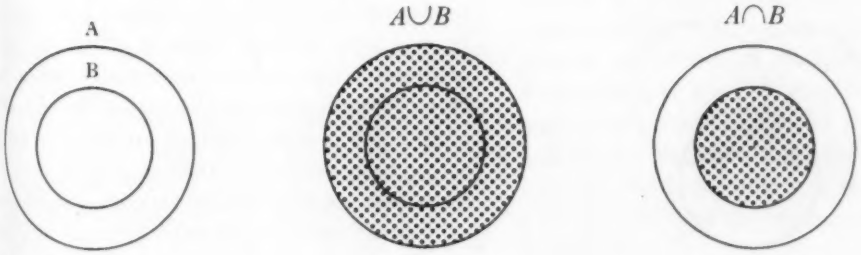
In this case $A \cup B$ would consist of all of set A and all of set B. $A \cap B$ would be the null set, $\varnothing$; that is, there are no elements common to A and B.

2. Sets A and B have some points in common:



In this case $A \cup B$ would consist of all the elements in A which are not in B, plus all the elements in B which are not in A, plus all the elements which are in both A and B. $A \cap B$ consists of all elements which are in both A and B.

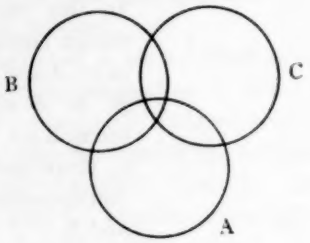
3. Set B is a proper subset of set A:



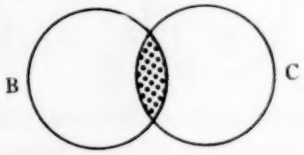
$A \cup B$ is set A.
$A \cap B$ is set B.

4. Here is an example which visually demonstrates union and intersection when there are more than two sets.

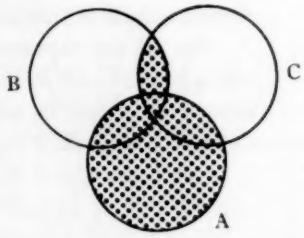
Given; three sets A, B and C all having some points in common:



Then $B \cap C$ is given by:

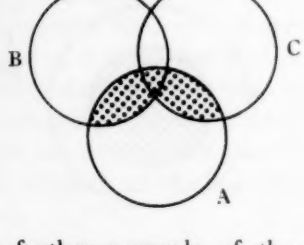


Now if we take the union of A with this intersection of B and C, we obtain:



The shaded portion can be expressed in symbols as, $A \cup (B \cap C)$.

Similarly we can represent the set $A \cap (B \cup C)$ by the shaded area below:



As a further example of the concepts previously explained let A be the set of closed polygons with four sides and let B be the set of four-sided polygons with opposite sides parallel. In this case every element of B is an element of A so that we may say that B is a subset of A. Furthermore, B is a proper subset of A because there exists four-sided figures which do not have their opposite sides parallel. The union of A and B is A; this is so because every element of B, having four sides, must be an element of A. The intersection of A and B is B; the elements common to both sets are only those with opposite sides parallel, namely the elements of B.

We have developed some of the basic concepts and vocabulary of sets, not the "algebra of sets" which is more detailed and abstract than there is space or need for here. On the basis of the very few definitions given we are now able to set forth possible suggestions for use of the notions of sets as a unifying concept in elementary mathematics.

Implications for Elementary School Mathematics

The introduction into the curriculum of new mathematics is a gradual development that usually takes place from the top down; that is, new concepts pioneered by the research mathematician are first introduced into the graduate level, then to the undergraduate level, next into the high school and finally to the elementary grades. It is the opinion of a number of educators that certain changes could take place in the elementary schools that would not only help to bring about a smoother transition into a modernized high school program but would have a very definite positive effect on the arithmetic curriculum. Essentially what is being suggested for the elementary level is an adaptation of some modern vocabulary and a few basic concepts. It is expected that a friendlier, more interested attitude would result toward the study of arithmetic, as well as the development of a stronger background which could avoid a great deal of the difficulties the student normally encounters in his later mathematical studies. A great deal of research and experimentation is yet to be done along these lines.

We will now consider certain phases of modern mathematics that could easily be introduced in the study of arithmetic that are perhaps best described as "basic concepts and vocabulary." But first a word of warning: let us keep in mind that merely to use the terminology in isolated instances without context to arithmetical principles, or without understanding of its present and potential usefullness, would, for the most part, be a waste of time. It is the responsibility of the teacher to keep abreast of the latest research applicable to her situation that would enable her to use such new ideas with conviction of their value, so that they may be meaningful to the students.

Sets

The concept of a set, as previously described, can be employed throughout the elementary years. Essentially the notion of set, mathematically, does not conflict with our psychological notion of what we mean when we speak of "collection of things." In the earlier years the teacher can introduce the word "set" into the children's vocabulary, making certain that the word is used in accordance with the accepted mathematical description. For example, the students can learn to speak of sets of books, sets of words, sets of numbers, etc. When it appears that they use the word correctly, the children might next learn to distinguish between sets; i.e., the set of books on the desk and the set of books in the closet, the set of numbers on the blackboard and the set of numbers on the paper, the set of numbers from 1 to 10 and the set of numbers from 10 to 20.

In later grades the set terminology should gradually become more sophisticated. We can speak of the set of all even numbers, the set of odd numbers, the set of pairs of numbers whose sum is ten, the set of pairs of numbers whose product is 12, the set of numbers that are divisors of 100, etc. When the students are ready the set notation symbols can be introduced. It is clear that the language of sets can be adopted to all phases of the arithmetic curriculum.

Let us look at a number of specific arithmetic examples:

1. Given the set $\{12, 132\}$. (a) Using the members of this set form a new set whose members are the sum of the two, the difference of the first from the second, the product of the first times the second and the quotient of the second divided by the first. (b) Form another set listing the addends, sum, minuend, subtrahend, difference, multiplicand, multiplier, product, dividend, divisor and quotient of part (a) in this order.

Solution:

(a) $\{144, 120, 1584, 11\}$
(b) $\{12, 132, 144, 132, 12, 120, 132, 12, 1584, 132, 12, 11\}$

2. What two sets of numbers do we use when writing mixed numbers?

Solution: The set of whole numbers and the set of fractions.

3. Here is a set of decimal fractions $\{.4, 2.5, .33\frac{1}{3}, .75, .125\}$. Form two new sets as follows: (a) The set of per cents which are equal to the members of this set using the same order and (b) the set of fractions, in lowest terms, which are equal to the members of this set in the same order.

Solution:

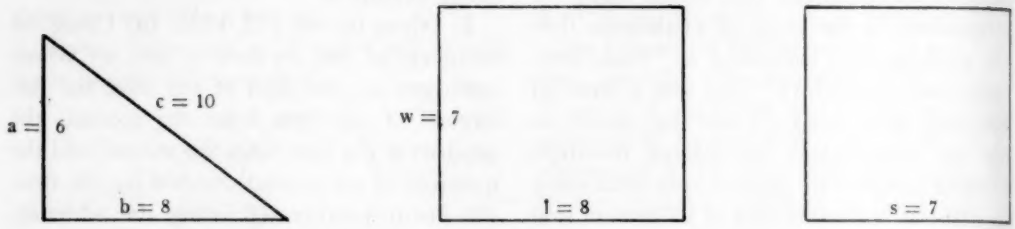
(a) $\{40\%, 250\%, 33\frac{1}{3}\%, 75\%, 12.5\%\}$
(b) $\{2/5, 5/2, 1/3, 3/4, 1/8\}$

4. (a) Form a set described as follows: The set is to have five elements; each element is to be a fraction equal to 2/3 but no two of the elements shall look alike. (b) Form two sets, the first consisting of all the numerators taken from the set in part (a) and the second consisting of the denominators. (c) Find the union of the two sets in part (b). (d) Find the intersection of the sets in part (b).

Solution:

(a) $\{4/6, 6/9, 8/12, 10/15, 100/150\}$
(b) $\{4, 6, 8, 10, 100\}$; $\{6, 9, 12, 15, 150\}$
(c) $\{4, 6, 8, 9, 10, 12, 15, 100, 150\}$
(d) $\{6\}$

5. Given a triangle, a rectangle and a square with dimensions as shown below:



(a) Give the set whose members are the formulas for the perimeters of these figures.
(b) Give the set whose members are the formulas for the areas of these figures.
(c) Find the union of the perimeters and areas of these figures, listing the members of this set from smallest to largest in number.

Solution:

(a) $\{(a+b+c),\ (2l+2w),\ 4s\}$
(b) $\{\frac{1}{2}ba,\ lw,\ s^2\}$
(c) $\{24, 28, 30, 49, 56\}$

Variable

The concept of variable is one which causes junior and senior high youth difficulty is not properly developed. Unknown number, literal number, and general number are all mixed in their minds to provide a very fuzzy concept of a variable.

We can begin early to ask such questions as: "How many more apples are needed to total 5 if we already have 2?" This progresses to some form such as: "I'm thinking of a number which when added to 2 gives 5. What is the number?" Or later, we state the same problem as $?+2=5$. or $__+2=5$. In the junior high school grades we present this in the form $x+2=5$. In each case we are dealing with a variable; we ask the student to replace the "?" or the "__" or the "x" with some member of a set. The variable is a *placeholder* for an element of a set.

Again, we may ask for the set of pairs of numbers whose sum is 10. Here we shall ask for replacements only from the set of integers; technically we say that the *domain* of discussion is the set of non-negative integers. The possible replacements then would be the elements of the following set: $\{(0, 10), (1, 9), (2, 8), (3, 7), (4, 6), (5, 5)\}$.

In work with formulas we develop early the statement "$P=4s$" for the perimeter of a square. Here the domain of the variable is the set of positive rational numbers; that is, we may replace "s" by any positive fraction or integer.

In contrast, the formula "$C=4n$" gives the cost, C, of n postage stamps at $4¢$ each. Here the domain of the variable n is the set of positive integers; that is, it makes no sense to replace n by a fraction as it is always necessary to purchase an integral number of stamps.

The concept of a variable as a placeholder for an element of a specified set of numbers is clear and unambiguous. This concept should be developed through the years in preparation for the detailed treatment of variables given in ninth grade algebra.

Development of the Number System

Early in the elementary grades we introduce youngsters to the set of positive integers. They learn to count objects by matching them with elements of this set. Formally we say that they place the objects in a *one-to-one correspondence* with a subset of the set of positive integers. For example, they count the fingers on one hand by placing their fingers in a one-to-one correspondence with the set $\{1, 2, 3, 4, 5\}$.

Our number system is extended at an early stage to include zero. This is necessary in order to be able to solve such problems as $5-5$. It also provides a number to associate with the null set. This new set $\{0, 1, 2, 3, 4, \cdots\}$ we shall call the set of

non-negative integers.

Later the youngsters are aided in the discovery of basic relationships among these sets of elements. That is, they learn that $3+2=5$. Furthermore we can ask, using appropriate language for their age level, for the set of pairs of numbers whose sum is 5; namely the set $\{(1, 4), (2, 3)\}$, when working with the set of positive integers, and the set $\{(0, 5), (1, 4), (2, 3)\}$, when working with the set of non-negative integers.

There are still many problems which cannot be solved using the set of non-negative integers, such as dividing 2 by 3. To allow us to solve problems of this type we introduce the elementary students to the set of fractions. The union of the set of non-negative integers and the set of positive fractions we call the set of non-negative rational numbers.

Again we search for relationships among the elements of the various sets of numbers and find, as one example, that the sum of two numbers does not depend upon the order in which they are added. That is, $2+5=5+2$, $1/2+3/4=3/4+1/2$, etc. In general, $x+y=y+x$ where x and y are placeholders (variables) for any elements of the set of numbers under discussion. Since this statement is true for all replacements of x and y we call it a law and it is given the name, *Commutative Law of Addition.*

A similar law is discovered for multiplication; $xy=yx$ for all replacements from our original set. This is called the *Commutative Law of Multiplication.*

An interesting and useful consequence of the adaptation of the commutative laws discussed above is the reduction of the number of addition and multiplication facts that are normally put to memory. That is, once it is learned that $2+5=7$, and the commutative law of addition has been established, it immediately follows that $5+2=7$, therefore making it unnecessary to memorize the latter addition fact. Clearly, this method reduces the number of addition facts considerably, and the multiplication facts would behave similarly.

In the junior high grades we find that our number system—the non-negative rationals—is inadequate for all the problems we meet. We cannot, for example, subtract 5 from 3. In order to make subtraction always possible we introduce the set of negative numbers, both integers and fractions. The union of the set of positive numbers, zero, and the set of negative numbers is called the set of rational numbers. With the elements of this set of numbers we are able to add, subtract, multiply, and divide and will always produce a result which is a member of the original set. This property is described by stating that the set of rational numbers is *closed* with respect to the four fundamental operations. As an illustration, when we take any two elements in our given set of rational numbers, say 2 and 5, and we add, subtract, multiply and divide them, obtaining 7, -3, 10 and $2/5$ respectively, we observe that each of the answers is still an element belonging to the set of rational numbers.

Further extensions of the number system take place later as we try to solve equations such as $x^2-2=0$, thereby introducing the set of irrational numbers, and equations such as $x^2+1=0$, thereby introducing the set of imaginary numbers.

From the early grades through the secondary school the mathematics we teach our students involves sets of elements and relationships among these elements.

Concluding Statement

Let us restate that the above considerations are merely an attempt at illustrating possible implications of the set notion for the elementary grades. Much research is necessary. However there is a good deal of information now available for the junior and senior high schools which can, for the most part, be extremely helpful to the elementary school teacher. Sources of this material are listed in the bibliography below.

Regardless of the research to be done or that which is already available the major responsibility, as always, lies with the teacher. In this respect we must honestly say that the initial effort on the part of the

teacher will be great; however, we are just as confident that the results will prove to be more than worthy of any such effort.

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EDITOR'S NOTE. Teachers need not be alarmed at the idea of "set" because set is merely a term applied to any collection of things that is presumed to have some common characteristic. Now the use of the term has been extended and mathematicians have made some interesting deductions about various sets. For the elementary school, we can talk about a set of numbers or a set of fractions giving the restrictions of a particular set and our pupils will soon grasp the idea and furthermore many of them will create their own sets. Placing a restriction or furnishing a description of a particular set has some of the elements of a puzzle or a charade.

Some teachers are becoming concerned lest the arithmetic of the elementary school become too abstract and "mathematical" and that it lose its significance in the social and economic affairs of children and adults. This need not be the case. It is always advisable to tie an operation in arithmetic to the social uses thereof so that people may realize that here is a body of knowledge that can be useful at all stages of development and is not something that people may learn just as an amusement. Many youngsters as low as the grades seven and eight have been intrigued by Venn diagrams. Perhaps this form of showing relationships can be applied even earlier. Children tend to want to "put things in their place" in the sense of classifying and identifying. They frequently have a keen sense of child logic. We can help them with this and frequently mathematical symbolism and the essence of mathematical logic can be applied.

BOOK REVIEWS

Figurets, J. A. H. Hunter. New York: Oxford University Press, 1958. Cloth, x+116 pp., $3.50.

This book contains 150 mathematical puzzles cast in the form of entertaining anecdotes. The author has included 16 Typical Solutions, illustrating the main mathematical concepts used, and a complete set of answers is provided. The purpose of the book is to provide fun for persons who are mathematically minded. This publication follows another of the same type called "Fun With Figures."

How did Pam and Peter do on a bicycle built for two? Did Jake save enough money to marry Janet? What about John and Jane's wedding present? These are questions you might like to answer if you like romance mixed with your mathematics. Or, if you like a logical approach to trickey questions on finance, you might like to find out the answers to such questions as: What became of Bill's nickel?, How much did Doug make from the partnership? Or, How about Bert's winnings at the racetrack? If you are the murder-mystery type reader, you may tingle your nervous system on a puzzler about "Crime After Midnight." Or, the poet may revel in jingles and figures of speech concerning a variety of subjects, each of which has a puzzle of some type to add spice to the poetic fancy. There are several calculations done in code for the puzzler who enjoys his puzzles straight with no extra trimming.

"Figurets" is a delightful little book for the adult layman who likes figures. It might provide some enjoyment for the more able adolescent who is familiar with the equivalent of elementary algebra. The mathematics teacher could find some helpful material for the "brain-teaser" section of the math bulletin board, or school newspaper. It would be a most suitable gift for that friend who likes to work with figures and who is about to go on a relaxing vacation or for that friend who is a shut-in.

INA MOE SILVEY

Conducting a Math Exposition

HAROLD KARBAL
Vernor Elementary School, Detroit, Mich.

LAST YEAR as the culminating activity of the Math Club we decided to hold a math exposition. Actually the spark had come from the Science Fair held in Detroit a short time ago. The club members needed some way to show what they had learned and done. The Math Club was composed of children with high mental ability, decided competence in arithmetic as evidenced on the Iowa tests, and proven interest in the subject, since they needed to be recommended by their teachers. The club members had studied in their weekly meetings number systems with base two and five and were currently learning to use the slide rule.

The idea of conducting an exposition was very stimulating to the group. Our first plans were devoted to how we would advertise the coming event. The artists, of course, made the posters to be posted in the hall. The others prepared short talks to be given each class during the morning conference periods.

The matter of awards demanded some attention. Those suggested varied from a slide rule to the grand prize winner to that of simple ribbons. The latter suggestion was the one finally used. Gold ribbons were given to first prize winners, blue ribbons to second, red ribbons for third, and white ribbons for honorable mention. The latter ribbon was actually given to all those whose exhibits were accepted for showing but not prize winners. Thus the day after the exposition all children who entered the contest had a ribbon to wear.

The entire school was divided into the following divisions:

Division I—Kdg. & Grade 1
Division II—Grades 2 & 3
Division III—Grades 4, 5, 6
Division IV—Grades 7 & 8

As feeling for the contest grew several teachers printed lists of possible entries.

From the beginning both parents and teachers were encouraged to help and direct. Reports came in daily of projects growing at home. We felt also that this was an opportunity for eliciting the active support of parents in a school project. It was understood that the children were to use their own ideas with the parents helping in construction, painting, lay-out and the like. This was an actual necessity with children in kindergarten and 1st grade.

The methods used by the teachers in stimulating worth-while projects were varied as:

"To stimulate my first grade class for the Math Exhibit I explained what an exhibit was and what kind of contributions could be made for it. We talked about our work with numbers and how we could show their meaning by making concrete objects."

"The children became interested through the talks presented by the Math Club member. We discussed various activities and projects that could be done. Different children told what they were planning to do."

"I used the desire to enter the contest which was sponsored by the Math Club, which some have hopes of joining in a higher grade."

"Talking about where we find and use numbers around the house—and one child bringing in a photo of herself at the gas station led to a wholesale response."

"I merely explained the plans for the exposition to the children and that if they wished to participate they could work on something at home with the help of their parents. I suggested some things they might do. As things were brought in we talked about them and this aroused the interest of others."

Not only were homerooms, where arithmetic is taught as a subject to be represented but all special classes such as science, social studies, gym, music, auditorium, art, industrial arts, and library were invited to contribute exhibits. Likewise the library became the center for all research with a special display of books. Not only were children's books placed there but also many teacher's professional books.

The following bulletin was issued to give suggestions for entries:

1. Entries may come from any child in school.
2. It should be some original contribution of the following type:
   a. Original solutions to problems.
   b. An explanation of why some generalization works.
   c. Models.
   d. Directions for performing out-of-the-ordinary procedures as you have discovered them.
   e. A study of some subject like insurance or stock market.
   f. Studies of ancient ways of working problems.
   g. Computers.
   h. Shortcuts in arithmetic.
   i. Any explanation of a special interest which is based on math: ham radio, electronics, etc.
   j. Experiments to prove the validity of some generalization.
   k. Correlation between arithmetic and another special subject.

Likewise other teachers prepared lists of worthwhile projects. A few sample items are given below.

1. Ancient methods of arithmetic.
2. Geometric design paintings.
3. Distance and time chart for reaching planets.
4. Heights of mountains.
5. Scale models of school, etc.
6. Problems in baking cake, etc.
7. Original mathematical games.
8. Distances on foreign road maps.
9. Booklets on topics such as ancient numbers, etc.
10. Unusual measures such as, cubit, ell, span, etc.

Each entrant filled out an entry blank like the following:

Vernor School Math Exposition
Entry Blank

Name
Section Grade
Title of Exhibit
Brief Description

As time went on enthusiasm in the exhibit grew as was evidenced by odd shaped boxes and packages carried to school or taken from the trunks of cars. An especially pleasurable sight was the bright smiling faces of youngsters sent to the principal's office to show off their creations with the evident approbation of their classmates and teacher.

At first we planned on a simple display in the hall. However, when we realized how far the idea had grown we took over the largest room in the school. This room was soon filled to overflowing with many different and unusual displays arranged by grade.

The arithmetic supervisor sent two assistant principals to our school to act as judges. The criteria for judgment were:

1. Originality
2. Amount of actual child work shown
3. Clarity of presentation
4. Neatness
5. Entry at child's grade level or above

The interest on the part of the children, parents, and teachers from schools in other parts of the city was keen. We planned on holding the exhibit one afternoon for guests and one morning for the children. We were forced to hold it open two days longer by the demand of parents and teachers.

This list of some of the exhibits may prove worthwhile for ideas for children in other schools.

Division I

KINDERGARTEN

1. Counting game—different ways of dividing five.
2. Counting scrapbook.
3. Number lotto game.
4. Clock face.

GRADE 1

1. Pocket chart for adding and subtracting game.
2. Counting rhymes picture book.
3. Counting down for rocket blast-off to help learning to count backwards. (Poster)
4. Counting by 2's. Train made of paper cups with spoons in each car.
5. Temperature changes shown on drawn thermometers. Record kept for three days.
6. Number story and three dimensional figures to show size and position down, up, big, little.
7. Adding numbers to make 10. Pipe cleaner ponies in circus rings.
8. Train showing counting by 5's.
9. Abacus—counting by thousands, hundreds, tens, and ones.
10. Barnyard scene showing various ways of grouping 6.
11. Counting frame with beads strung on wire,
12. Abacus—100 beads in strings of 10 on a wire frame.
13. Number story.
14. Numbers at home showing photographs taken of everyday scenes of children: gasoline station, reading address, telephone number, calendar, temperature, TV channel.

Division II

GRADE 2

1. Puzzle book.
2. Number facts board.
3. Models: telephone, speedometer, thermometer.
4. Number wheel—counting by 5's and 10's.
5. Fractions in pictures (poster)
6. Fun with numbers—chart.
7. Making a potholder.
8. Drawing a house with ruler.
9. Various games.
10. Astronomy book.
11. Large wooden clock to show time and fractional parts of $\frac{1}{2}$'s and $\frac{1}{3}$'s.

GRADE 3

1. Games.
2. A quipu made with shoe strings.
3. Cardboard slide rule.
4. Roman counting.
5. Number stories around the world.
6. Six hours around the world with clock faces showing time.

Division III

GRADE 4

1. Posters and games.
2. Foreign currency.

GRADE 5

1. Foreign coins with U. S. values shown.
2. Early number chart.
3. Abaci.
4. Trading without money.
5. Protractor transit.
6. Clay figure of man showing how ancient measurements were formulated.
7. Games and puzzles.
8. Napier's bones.
9. Graphs.
10. Model furniture for planning a room.
11. Six methods of checking subtraction.
12. Hand method of multiplying.

GRADE 6

1. We count calories.
2. Measuring for grain of fabric in dressmaking.
3. Piano Education.
4. A trip for 2 by plane to Miami, Florida. Comparison of round trip with fare for one way.
5. Flip flop circuits based on binary scale.
6. Measurements chart of long ago.
7. Pendulum to show time changes.
8. Measuring and counting calories for a reducing diet—actual child used for experiment.
9. Model rocket made to scale.
10. Short cuts to arithmetic.
11. Greek number system.
12. Diagram of a one tube radio with actual model.
13. Model of Washington Monument.
14. The problems of running a newspaper.
15. Model ranch house with complete floor plans.
16. Tower of Hanoi.
17. Booklet on foods and numbers.
18. Model bridge with scale drawing.
19. Tropical fish display with scrapbook.

Division IV

GRADE 7

1. Enlarging pictures through scales.
2. Scale model of house.
3. Earth satellites.
4. Abacus.
5. Sun dial.
6. Geometric figures and forms.
7. Model of Golden Gate Bridge. (to scale)
8. Model of Mackinac Bridge. (to scale)
9. Graphs.
10. Family budget.
11. Comparison of avoirdupois and apothecary weights.
12. Proving formula for measuring heights.
13. Display of geometric design in everyday objects—Model of a street corner.

GRADE 8

1. Incubator with fertile eggs, various graphs kept.
2. Transit.
3. Binary system of numbers.
4. Micrometer and how to use it.
5. Model home with scale drawing.
6. Quinary system.
7. Comparison of Metric and Linear system.
8. Quadrant.
9. Comparison of highest mountains in each continent.
10. Geometric mobiles.
11. Calendar for 55 years from 1943–1997
12. Scale model of Mackinac Bridge.
13. University mileage chart.
14. Cubic measurement shown with blocks.
15. Map showing mileage to take tour of U. S. with problem sheets.

AUDITORIUM

1. Arithmetic plays.
2. How we use arithmetic in plays.

HOMEMAKING

1. Dresses—comparison of homemade clothes with bought ones.
2. Weights and measures of foods commonly used.
3. Amount of different materials used in making dresses depending on various widths in materials—chart form.

GYM

1. Model games.
2. Scoring of different games.
3. Booklet on how to figure baseball averages.

ART

1. Chart showing proportion.

LIBRARY

Various reference books.

MUSIC

1. Dictionary.
2. How to use fractions with notes.
3. Piano education graphs.

SOCIAL STUDIES

1. Convention Hall—comparison of its size.
2. Reference measures chart to help understand math statements found in Social Studies books.
3. Scale drawing of Michigan.
4. Graphs and charts showing Michigan as a leader.

SCIENCE

1. Ham radio set up.
2. Diagram of atomic reactor generating electricity.
3. Time and distance chart of radio waves and rockets to the planets.
4. Solar system chart.
5. Scale diarama of planets in solar system.
6. Height of mountains.

What was the evaluation of the teachers of our school? These were typical.

"The value of the Math Exhibit was two fold. It showed that the children had a good understanding and workable knowledge of arithmetic which they demonstrated by the projects they made. It gave them a feeling of satisfaction and accomplishment to see their work displayed. Their parents got a good concrete example of how the children learn their work and can apply the knowledge to a concrete example of work."

"This project proved most worthwhile in bringing out Arithmetic and Math interests. It stimulated pupils to do creative and individual work. The results were amazing; and the display was excellent."

"The exposition was valuable in the respect that it aroused a new interest in arithmetic on the part of the children and it provided an opportunity for the children and their parents to work together on a school project."

"The response to the exposition was excellent; interest ran very high. I think the thing that pleased me most was the fact that it was not only the good math students who participated. Several projects, some very good ones, were brought in by students who ordinarily show little interest in the math period of homeroom."

"Personally, I thought the exposition was terrific. The student's work was good and the layout of the show itself was very attractive."

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EDITOR'S NOTE. Why not hold a Mathematics Fair? The values come chiefly from how such an event is stimulated, guided, and conducted. The Vernor School took the project seriously and apparently achieved a great deal more than extending the interest of pupils in mathematics. It is good to note that pupils as low as the kindergarten were included. It is amazing what interests and abilities some pupils possess and frequently these are pupils that the casual teacher has not identified. Frequently it is a simple little discussion that serves to start a youngster on a thought process that leads him into a new mental adventure. A Fair can become a great deal of mere "busywork" or it can be made something of high significance to the pupils and the school patrons. Perhaps other schools will hold the Fair type of open house and will thereby stimulate their pupils as was done in Mr. Karbal's school in Detroit.

I Like to Do It This Way

THE ARITHMETIC TEACHER would like to print more contributions from teachers in which they tell how they do some particular thing in connection with children learning arithmetic. This may cover all ages and all phases of school life in which some quantitative factor is involved. Please send these contributions to Dr. Joseph J. Urbancek, Associate Editor, 1112 Grant Street, Evanston, Ill.

Linda Learns the Hexal System

RUTH T. DEERY
Longview, Washington

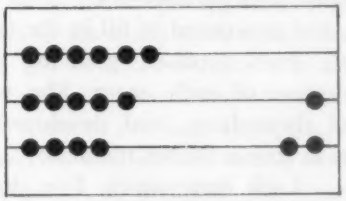
THE BEST WAY to gain understanding of English, it is said, is to study another language. Only by learning to think in another language can a person view his native tongue with something resembling detachment. Similarly, one way to gain understanding of our decimal number system should be to study a number system with a different base. Then it becomes clear that our decimal system is not the basis of the universe at all, but a tool for understanding the universe, a tool invented by the human mind, and one of many possible systems for organizing size and number.

Bright-eyed Linda, nine years old and in the fourth grade, tells me she plans to be a mathematician. When I suggested that she might learn something new and interesting about numbers and help me with a project besides by studying the hexal number system with base 6 and making a hexal abacus, she wanted to start right away, that very minute, before I shaped the loaves of bread for the pans.

There are, of course, many possible number systems. The duodecimal system, with eleven digits and zero, is thought to be the best for all-round use because computation with commonly-used fractions would be very easy in a system whose base is evenly divisible by three and four as well as by two. However, this system necessitates the use of two new symbols for our ten and eleven, and it seemed to me that this presented an unnecessary complication. My purpose was to give Linda a new perspective on our decimal number system. Since this could be done as well with a system based on a few digits as with one based on many, I chose what seemed the second best system from the point of view of working with fractions—the hexal, in which 10, (5+1 units), is evenly divisible by 3 as well as by 2.

Linda listened to my explanation of the new system only long enough to understand what would make the hexal abacus different from the one she ordinarily used. She hurried off to collect materials—wooden baby beads, an old box, stout string—but, alas! there were not enough beads of one color. She couldn't bear to wait to make her abacus till her daddy could drill holes in some blanks. Mixing colors she seemed to consider a desecreation, so I suggested a quadral abacus as an interim project. She made one quickly and begged for problems. It apparently never occurred to her that we should change our main project to a quadral one, for she waylaid her daddy the minute he came in the door and insisted that he drill the blanks immediately, and she made the hexal abacus forthwith.

Hexal Abacus



4+4=12

For half an hour or so I gave her problems in addition and subtraction to work on her hexal abacus, and I checked her answers laboriously by converting to decimal numbers, doing the computation mentally and converting back to hexal numbers by means of a scale I wrote down for the purpose. She made few errors, and those she quickly corrected. Our session was interrupted before Linda was ready to stop.

Of course, it doesn't take as deep understanding of a system to work problems on an abacus as to work them mentally. The second day our problems were abstract ones in multiplication and division. First came mental problems with one-figure multipliers and divisors. Linda's methods spoke well for her arithmetic training in school; for instance, she found half of 120 by thinking one-half of 100, or 30, plus one-half of 20, or 10. Thirty plus ten equals forty; half of 120 is 40.

After about ten minutes and ten mental problems, we turned to pencil and paper problems with one and two figure multipliers and divisors. Now we both felt more at home in the system. Linda made only one error, and I checked the answers by inspection instead of resorting to my conversion scale. It seemed painfully slow, however, and I never trusted my answer till I had done the problem twice. We did only twelve problems in half an hour.

At our third session, I presented Linda with two charts to fill out, one for multiplication and one for conversion. For the multiplication table I filled in the first row and the first entry in the second and handed it to her. She tried first to figure out some system in the entries which would eliminate the need for thinking out each product individually, gave up after discovering several errors, and proceeded to fill in the table by thinking each product, glancing up for confirmation of each entry. She was interested throughout, and developed some insights as shown by her remarks: "2, 4, 10, 12, · · · Look how much I've done already! 3, 10, 13, · · · that's like our fives. 4, 10, 12, · · · this is harder, let's do 5's first. 5, 14, 23, · · · Oh, like our nines! 32, 41, 100? No, 50, 55. Now, 4 times 5 · · · where's 5 times 4? · · · 32." It took her fourteen minutes to fill out the table, and she did it all mentally. The transfer of insight from the decimal system is obvious.

HEXAL MULTIPLICATION TABLE

1	2	3	4	5	10
2	4	10	12	14	20
3	10	13	20	23	30
4	12	20	24	32	40
5	14	23	32	41	50
10	20	30	40	50	100

Having Linda make a conversion table, based as it is on simple counting by ones and an understanding of place value, seemed a good opportunity to see if Linda's apparent understanding of the hexal system, in which she had worked problems using all four processes, and previously of the quadral system, in which she had worked a few problems using addition and subtraction, transferred at all to another system totally new to her, the binary system, based on one digit and zero.

I had labelled the columns, filled in two squares, named the quadral column ("Quadral? Is that 5's? Oh, I know, quadruplets"), and told her that the binary system is based on two so that our 2 is binary 10 and our 3 binary 11. The decimal and hexal columns she filled in confidently and without seeking confirmation. The quadral column she filled in almost as quickly and asked for confirmation only after it was

NUMBERS ON FOUR DIFFERENT BASES

Decimal	Hexal	Quadral	Binary	Decimal	Hexal	Quadral	Binary
1	1	1	1	10	14	22	1010
2	2	2	10	11	15	23	1011
3	3	3	11	12	20	30	1100
4	4	10	100	13	21	31	1101
5	5	11	101	14	22	32	1110
6	10	12	110	15	23	33	1111
7	11	13	111	16	24	40	10000
8	12	20	1000	17	25	41	10001
9	13	21	1001				

completed. On the binary column, she asked, "1, 10, 20?"

"No," I said, "no 2's," and she filled out the table slowly with a number of erasures, asking for confirmation four times, making one error that she didn't discover herself. Most of the eight minutes she took to fill out this table was spent on the binary column. When she finished, she said, gleefully, "Mother, our 17 is ten thousand one!"

Several days later I had an unexpected opportunity to test Linda's understanding of the structure of the hexal system, as opposed to her ability to work problems in it. She was playing with the Stern unit box which I had brought home for the benefit of her little brother. I asked her if she could see why it could be called a hundred-square. She could, after brief explanation. Then I asked if she could make a hexal "hundred-square," and she did so without hesitation. I asked her how many cubes she would find in her hexal hundred-square if she were to count them in the hexal system and how many if she were to count them in the decimal system. She answered, "Uh . . . one hundred," and "Uh . . . thirty-six," without doing any counting. Since we had not been discussing the hexal system for several days, I gathered that the idea had taken root.

We didn't do much with fractions. At school Linda has studied only addition and subtraction of simple fractions, so my principal reason for choosing the hexal system for this project—ease in computing with fractions—proved to have been irrelevant. She did the first set of four problems with amazing speed, and, as I was checking them slowly, feeling a little confused, she remarked that she had figured it so quickly by using the decimal system. For instance, 1/5+3/5+4/5=decimal 8/5=1 3/5 in either system. When I said, "Then you weren't thinking in hexal terms," she answered, "No, but I can go back and do them again that way." She did so, leaving the fraction unreduced. In the above example, for instance, the result was 12/5.

I had not foreseen that the small numbers which it seemed best to use in our fractions session could be worked in either system and the results would be identical as long as the fraction was reduced. Having discovered this, I had Linda spend the rest of this session reducing larger improper fractions such as 22/5=2/4/5, with the numerator above decimal 10 to encourage solution in the hexal system instead of by converting.

I would have liked to take up the hexal point which corresponds to our decimal point with Linda, but, as she hasn't had the decimal point in school, I thought it better not to. Thus far her work in the hexal system did not seem to confuse her thinking in the decimal system, but it seemed to me that a brief exposure to a new arithmetical concept in the hexal system in advance of her exposure to it in the decimal system might very well prove to be a stumbling-block. Also, it seemed wiser to have her introduction to this new idea be under the supervision of the teacher who would be carrying through for the entire year.

For our next session, I prepared a sheet of simple mixed problems, including story problems, to find points that might need clearing up. Linda did the problems without difficulty, but, when we scored the sheet in decimal numbers and I jokingly asked what her hexal score would be, she puzzled a long time and needed several hints.

Following the surprising discovery that, although Linda could work within the hexal system, she couldn't readily convert even small numbers from one system to the other, I wrote a dozen numbers in each system to be converted and showed her a method of using counters to facilitate the conversion. She was to set up the number to be converted, using large counters for "tens" and small for ones in either system; then, mentally she was to change the value of the tens counters and call them sixes to convert from decimal to hexal numbers, or fourteen to convert from hexal to decimal, as follows:

$h\,22 = d\,(?)$ $\quad h\,⑩\,⑳\,○\,○ = d\,⑥\,⑫\,○\,○$

$d\,31 = h\,(?)$ $\quad d\,⑩\,㉘\,○ = h\,⑭\,㉜\,㊿\,○$

$h\,22 = d\,14$

$d\,31 = h\,51$

This was not meaningful for Linda at all. She made many errors and became hopelessly confused, although she was still interested enough to want the problems checked and to correct her errors. The working, checking, and correcting of the two dozen numbers took 40 minutes, and at the end of that time she didn't seem to comprehend the process any better than she had when we started. I asked how she thought we might clarify the relationship for her; perhaps by using counters of different colors, as *d*ark for *d*ecimal tens, *h*ot pink for *h*exal tens. Linda had a better idea: use large counters for decimal tens, small ones for hexal tens, and counters of an entirely different shape for ones, to be used alike in both systems.

So, at our next session, we got out the big blocks to represent decimal tens, the little blocks to represent hexal tens, and bottle caps to represent ones. Before doing problems, we set up the basic conversion facts:

d 10 = *h* 14 □ = □ ○ ○ ○ ○

h 10 = *d* 6 □ = ○ ○ ○ ○ ○ ○

d 30 = *h* 50 □ □ □ = □ □ □ □ □

Changing decimal to hexal numbers was still difficult for Linda, and it took her 40 minutes to do the first dozen problems, the same dozen that she had stuggled with the previous day. This time, however, she corrected her own errors quickly and with comprehension when they were pointed out to her instead of bogging down in a morass of incomprehension as she had before. The other dozen problems—the hexal-to-decimal—she did in five minutes, after the first few, skipping the step of changing hexal for decimal blocks. She found it easier to count by "decimal" sixes. We then went back to the same dozen troublesome decimal-to-hexal problems. This time the light seemed to dawn, and she did the entire set in about ten minutes, still using the two sizes of "tens" counters, however, as counting by hexal "fourteens" is trickier than counting by decimal sixes.

It came as a surprise to me that conversion out of the decimal system should be more difficult than conversion into it. If I were doing this again, with the benefit of hindsight, I would take up the processes separately, being sure hexal-to-decimal conversion was mastered at least with the counters before introducing demical-to-hexal conversion.

Diagnostic Test in Hexal Arithmetic

4	13	22	13	4	3	10	14
+4	−5	−3	+12	×5	×4	×3	×3
12	4	14	25	32	20	30	50

Joan had two sets of jacks. Jean had twice as many as Joan. How many sets did they have all together?

2
+4
10

Reduce:

$\frac{20}{3} = 4$ $\frac{14}{5} = 2$ $\frac{21}{3} = 4\frac{1}{3}$ $\frac{13}{3} = 3$

4 r1 3 r1 15 3 r2
4)25 3)14 2)34 5)25

Neddy has three cars and a truck. Kevin has one car and two trucks. How many vehicles do they have all together?

3
1
1
2
11

Change these decimal numbers to hexal numbers:

d	*h*	*d*	*h*	*d*	*h*
27	= 43	29	= 45	59	= 135
42	= 110	46	= 114	28	= 44
13	= 21	37	= 101	25	= 41
31	= 51	72	= 200	30	= 90

Change these hexal numbers to decimal numbers:

h	*d*	*h*	*d*	*h*	*d*
43	27	45	= 29	31	= 19
22	= 14	54	= 34	35	= 23
100	= 36	32	= 20	23	= 15
34	= 22	12	= 8	25	= 17

Now, thought I, Linda was probably ready to convert with pencil and paper, so I worked out what seemed to me an easy method of factoring, a method which was the equivalent of conversion with counters. At our next session, I showed her how to divide the decimal number to be converted by six (hexal ten) or the hexal number to be converted by fourteen (decimal ten), convert the divisor and quotient mentally to the other system, multiply them, and add the remainder. She could use the formula, but very slowly, with many errors, and without much comprehension of the meaning of the steps despite my repeated explanation of the theory and comparisons with the concrete experience with counters. She obviously wasn't ready for pencil and paper conversion, so I decided to scale down the project by asking her to convert only those numbers that she could easily figure mentally (up to a dozen or so) or work out with a few blocks (up to hexal 200). I didn't want to insist on any learning that was so difficult that she would feel discouraged. As it was, she kept coming back asking for more hexal problems, and this enthusiasm was too precious to jeopardize.

Linda had been wondering what she would do at the approaching demonstration of the hexal system for the group of teachers in the arithmetic methods class, and so had I. So we sat down and worked it out together at our last session. I gave her several simple problems in each category; she worked them aloud; and I selected the most interesting ones and decided what points to bring out with each. We both enjoyed the demonstration itself.

This seemed the logical conclusion of the project. I could think of nothing further to do with it except drill her, a poor idea, it seemed to me, as I certainly didn't want her to go through life with a sneaking suspicion that four and four are twelve in everyday concerns. I might have given her more complex problems or ones using larger numbers (of doubtful value at best, as meaning and not pencil and paper facility was the object), or we could have gotten ahead of her decimal learning in school (unwise, I decided, for reasons detailed above).

This project seemed to me a great success on three counts: 1) Linda learned something interesting about arithmetic that she might otherwise have missed; 2) I learned something about the way her mind worked and how this differed from what I had expected; 3) and we both found it a delightful mutual project for mother-daughter sharing.

EDITOR'S NOTE. Mrs. Deery is a housewife who, while in an arithmetic workshop, developed the ideas with her daughter as they are described in the article. Both mother and daughter have a good deal of understanding of the essential features of a number system and the role of the "base" of a system. It is necessary to keep the base in mind as one reads the article. Symbols might be misleading to a casual reader. For example, 100 might mean one hundred single things in base ten, 36 base-ten things in base six, or 8 base-ten things in base two. What shall we call the symbols *100* on various bases? At a later stage when pupils have learned about powers and exponents a discussion of number bases can become more sophisticated.

The editor suggests the use of an open-end abacus made by placing rods in a base piece of wood. Such a frame can be used for any base system and has the advantage of permitting more single counters on a rod than the base being used and thus giving more significance to the role of base.

How Much Time for Arithmetic?

G. H. Miller
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Not too many years ago, students in elementary schools were taught arithmetic for one full period a day in all grade levels. Arithmetic was considered one of the major goals of education. One only needs recall the popular solgan "The Three R's" (reading, 'riting, and 'rithmetic) to remember the influence that arithmetic had in our elementary curriculum since the founding of our country.

One of the more potent influences in changing this pattern was a number of studies by Washburne and the Committee of Seven. These men made extensive experiments in schools in Illinois and other states to determine the best time to introduce the specific topics in arithmetic. The procedures and results of these investigations were reported in the Twenty-Ninth Yearbook (1930) (7) and the Thirty-Eighth Yearbook (11) of the N.S.S.E. and in other articles by Washburne (6, 8, 9, 10). Washburne and the Committee of Seven pointed out that there was an optimal mental age when the teaching of the subject would produce the best results. The instructor could thus prevent many pupil-failures in arithmetic. They recommended that the curriculum in the elementary schools be revised so that the topics could be introduced at the time when the student would be most likely to master the material. These are several of the recommendations:

Topic	*Minimal Mental-Age Level*
Addition Facts	
Sums 10 and under	6 yr. 5 mo.
Sums over 10	7 yr. 4 mo.
Multiplication Facts	8 yr. 4 mo. or 10 yr. 2 mo.
Percentage, Case I	12 yr. 4 mo. or 13 yr. 4 mo.

There were several educators who wrote articles pointing out certain deficiencies of the study. These men were Brownell (1, 2) and Rath (4, 5). Several of their criticisms were:

1. There are inadequacies of certain of the tests used in the experiment, i.e. the subtraction test was too difficult.
2. There was an exaggerated emphasis of the interrelationship between the mental age and the topic placement in arithmetic.
3. The theories of readiness which the committee proposed were inconsistent.
4. Even though the authors admitted many limitations of their studies, they still made recommendations on the basis of their results.

Despite the contentions of Brownell and Rath, the recommendations of the Committee of Seven were accepted as a major criterion in grade-placement throughout the nation. In fact, an analysis of the present elementary textbooks made by the author shows their placement very much like the data provided by the Committee of Seven.

Another powerful influence in the de-emphasis of teaching arithmetic was the Progressive Education Movement. During the time of the dominance of the progressive educators, recommendations were made which reduced the stress on arithmetic and placed the major emphasis upon social studies. The slogan "instructors teach children, not subject matter" led to practices which curtailed the amount of time devoted to arithmetic. Instead, there were integrated courses, core-curriculum courses and child-centered courses. According to this theory, the teachers were able to instruct arithmetic more successfully through combined activities in which arithmetic was used to compute the cost of articles while playing storekeeper, to find the cost of candy while having a taffy pull in class (3) and other similar activities.

Thus, in the last thirty to forty years, arithmetic played a minor role in the elementary curriculum.

With the advent of the Korean conflict and the cold war, the need for more engineers, mathematicians and scientists was emphasized by those in industry and in the armed forces where these shortages of personnel caused considerable concern. In order to prepare more scientists, many schools changed their curriculums and provided more mathematics. However, this increase was mostly noted in the high schools. Now with the arrival of the Sputniks, the race to the moon, and the struggle for scientific superiority, the need for high quality training in mathematics is well recognized by many individuals.

With all the changes that have been made in the elementary curriculum, it is desirable to determine the amount of time which is devoted at present to arithmetic in the elementary grades. It is the elementary school which should provide the basic foundations in arithmetic for the students in mathematics and the sciences. These foundations must be strong so that our students will obtain a better education in these areas. This study was devised to find the amount of time which is devoted to arithmetic in each grade level of elementary school, and to make recommendations based on the conclusions.

Survey of Time Spent

A form was sent to school systems all over the country. The supervisors of elementary education, principals, and instructors, were asked to indicate the daily amount of time which was allotted to arithmetic in each grade level. Also they were asked to indicate the basic content taught at these grade levels, if time permitted. A total of 34 forms were received from large school systems, and 44 forms were received from the small school systems. Over half the forms received from the small school systems were from cities in Illinois.

The results of these daily time schedules were tallied and the median for each grade level was determined. The per cent of the schools allotting a specified time for arithmetic was also computed for each grade level in intervals of ten minutes. Since there were an insufficient number of forms returned which had the curriculum content written on them, that analysis will not be considered in this paper.

The Findings

The following Tables show the results of the different grade levels in time allotment for arithmetic for the large and small school school systems.

Conclusions

An analysis of the tables shows the following results:

1. The amount of time spent in arithmetic ranged from a median score of 23 minutes per day to 45 minutes per day in the large city schools and 30 minutes per day to 47 minutes per day in the small school systems.
2. There is a marked difference in the amount of time spent in the lower grades (1st, 2nd, and 3rd). The lower the grade, the less the time devoted to arithmetic.

TABLE I

TIME ALLOTMENT FOR THE LARGE CITIES

Grade	Median	Number of Minutes per Day						
		0–9	10–19	20–29	30–39	40–49	50–59	60–69
1st	23	14%	24%	39%	14%	6%	3%	
2nd	32	3%	3%	38%	50%	3%	3%	
3rd	40			3%	47%	29%	18%	3%
4th	45				3%	76%	15%	6%
5th	45				3%	70%	21%	6%
6th	45				6%	67%	15%	12%

TABLE II

TIME ALLOTMENT FOR THE SMALL CITIES

Grade	Median	Number of Minutes per Day						
		0–9	10–19	20–29	30–39	40–49	50–59	60–69
1st	30	5%	9%	36%	25%	13%	7%	5%
2nd	35			25%	46%	11%	9%	9%
3rd	42			7%	36%	36%	10%	11%
4th	47			2%	11%	53%	21%	13%
5th	47				11%	57%	19%	13%
6th	47				11%	55%	23%	11%

3. Very little difference is noted in the median time allotments in the upper elementary grades (4th, 5th, and 6th).
4. A similar pattern in time allotment is observed for the small and large school systems. However, there is a difference of two to seven minutes between the medians of the small and large schools. This indicates that the small school systems spend more time in arithmetic than do the large city schools. This condition is especially true in the lower grades.
5. There is a wide variation of over sixty minutes in some cases between the time allotments within each grade level in the different schools. The lower the grade, the greater the variation.

It is interesting to note that all the time allotments for the instruction of arithmetic except one (1st grade in the large cities) exceeded the amount of time spent by the schools cooperating in the studies made by the Committee of Seven. Thus, most school systems have increased the amount of time devoted to arithmetic over that stipulated by the Committee of Seven.

The great variation in the time spent in arithmetic could lead to many difficulties. For example, if a student transfers from a school in which little arithmetic was taught to one which emphasizes arithmetic, he would be at a serious disadvantage in comparison to the students in his new school. Or in the case where a student transfers from a school where he is taught a lot of arithmetic, he may become disinterested in arithmetic because of continued repetition. In either instance there exists the possibility that the achievement of the student in arithmetic may be impaired.

Even if the student remains in the same school system, the deficiencies in his arithmetical background may arise when he goes on to junior high school, high school, or college and has to compete with students who have a superior arithmetic background.

Recommendations

After analyzing the results of the study, the following recommendations are made:

1. The length of time for the teaching of arithmetic in the elementary grades should be made more uniform throughout the nation's schools. This condition would equalize the mathematical opportunities of the students throughout the nation.
2. In view of the greater need for mathematics in our society, it might be well to consider increasing the time allotment in arithmetic, especially in the lower elementary grades of those schools where little time is devoted to arithtic.
3. A study should be made to determine the content of arithmetic that is taught in each elementary grade to discover the similarities and differences in instruction of arithmetic in the schools throughout our nation.

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EDITOR'S NOTE. How much time for arithmetic instruction in the elementary school? More important than time alone is the way in which the time is used. Dr. Miller points to the influence of the old Committee of Seven some 25 to 30 years ago. Many of us then were concerned with the pronouncements of this committee because we could learn little about the modes of instruction and the views of the committee toward the content of arithmetic and its role as a school subject and as a factor in our society. Frequently results were based upon small numbers of cases and on an arithmetic that leaders in the field were not supporting. In more recent years we have had an increased emphasis upon "stretching topics" or introducing the simpler concepts and ideas at an earlier age and delaying the mastery stage of computational work. By one method or another we do want our pupils to learn arithmetic and that is an arithmetic that features thinking and understanding of both the arithmetic and its significance in our society. How much time should this take? We do not know but we tend to agree that we should spend the time necessary to achieve good results.

We are thankful to Mr. Miller for providing the tables which show the prevailing practice in the sample of schools covered in his report. It would be interesting to know whether the schools that spend more time from grades one through six are those that achieve better results.

Teaching Signed Numbers in Grade 8

ALEXANDER CALANDRA
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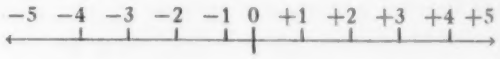
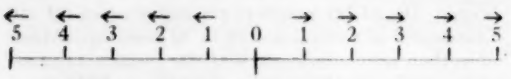
DURING THE PAST YEAR the writer had the opportunity of experimenting with the teaching of a course involving both science and mathmatics at the elementary school level. One of the experimental techniques which turned out to be very helpful was the introduction of a different notation for signed numbers. The method used is recommended to teachers who teach some algebra at the eighth grade level.

Signed and directed numbers were introduced in terms of positions on a directed line in the usual way, but the symbols for the signed numbers consisted of numerals with small arrows above them. Thus the convention $+3$ was written $\overrightarrow{3}$ and the convention -3 was written $\overleftarrow{3}$, the notation being essentially a direct application of the picture of a directed line:



which can be written:



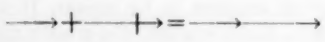
This notation has the advantage of not confusing the original meaning of plus [+] with the process of going to the right.[1]

The operations corresponding to the symbols [+] and [−], when they involve [signed or directed] numbers were given the following definitions:

[+] is defined as "followed by" in the same sense indicated by the equation:

$$\overrightarrow{2}+\overrightarrow{3}=\overrightarrow{5}$$

which is the equivalent of the graphic solution:



[−] is defined as "followed by with the direction reversed," thus the equation:

$$\overrightarrow{2}-\overrightarrow{3}=\overrightarrow{2}+\overrightarrow{3}$$

therefore:

$$\overrightarrow{2}-\overleftarrow{3}=\overrightarrow{5}$$

which is commonly written:

$$[2]-[-3]=[+5]$$

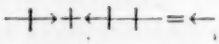
Another example of this process of subtracting signed numbers is:

$$\overrightarrow{2}-\overrightarrow{3}=\overrightarrow{2}+\overleftarrow{3}$$

therefore:

$$\overrightarrow{2}-\overrightarrow{3}=\overleftarrow{1}$$

or graphically:



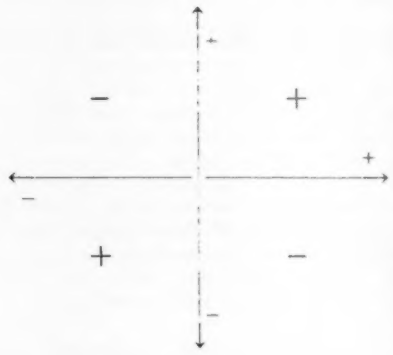
[1] It is interesting in this connection to note that the Egyptian hieroglyphic symbol for plus is $\wedge$ and that the corresponding symbol for minus is $\curvearrowright$, the meaning of the former symbol is that of a pair of feet walking toward and the latter symbol represents a pair of feet walking away.

which in the usual notation is:

$$[+2]-[+3]=[-1]$$

After the student has had some familiarity with the use of this new type of notation, he is shown the more conventional notation and is led to develop the conventional rules for the addition and subtraction of signed numbers.

In teaching multiplication and division of signed numbers, we have resorted to using directed areas in the same way that directed lines are used to introduce the addition and subtraction of directed numbers. In this approach the areas in the usual quadrants in the system of rectangular coordinates are given signed values and the product of any two numbers is treated as a directed area.



Although the directed areas could be given notations in terms of arrows, this system is much too cumbersome for elementary use. No attempt is made to interpret the convention but is used as a matter of definition.

Experimentation along these lines was made possible in part by a contract with the Cooperative Research Division of the U.S. Office of Education demonstrating the value of a combined science and mathematics course. The writer will be happy to correspond with teachers interested in using his approach.

Psychology Applied to the Teaching of Arithmetic

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IN THE LAST TWO DECADES numerous studies and experiments in the field of psychology have given us many sound principles of learning which are readily applicable to the teaching of all subjects. These principles have no doubt been successfully used by many teachers; however, to the best of the writer's knowledge, no organized statement of these principles of learning as applied to the teaching of arithmetic has been published. The writer will attempt in this article to draw together some of these principles of learning and then apply them to the teaching of arithmetic.

The principles of learning and their practical applications for the classroom teacher are as follows:

1. LEARNING SHOULD BE ORGANIZED AND MEANINGFUL

Our number system is a logically organized and closely knit system of ideas and interrelationships which can readily be made meaningful to the learner. The number system is based on the following simple principles: (1) each place or position has a value of some power of 10, (2) there are 9 digits and zero which indicate the frequency in any position, (3) zero can be considered a "place holder" in a number, and (4) the ratio of place values successively from left to right is 10 to 1 and from right to left is 1 to 10. These principles should be used in teaching all fundamental operations since they provide the child with an organized and meaningful method for putting together, for taking away, and for comparing numbers.

A child who understands positional value will know that the number 428 is 4 hundreds, 2 tens, and 8 ones; or 42 tens and 8 ones, or 428 ones. An understanding of the function of zero as a place holder will make it clear to the child that in the number 206 the zero indicates that there are no tens and if the zero were omitted, the number would become 26. In such a number as 5555 the child should understand the first five is ten times as large as the second five, that the last five is 1/10 as large as the third five, etc.

In working arithmetic computations, ten should be used as a standard group or reference point; for example, the child who in adding 7 and 5 thinks, "7 and 3 are ten, and 2 more make 12," or the child who solved 9×6 by thinking, "10 times 6 is 60, minus 6 is 54," or the child who solved $165 \div 3$ by saying, "There are 50 threes in 150 and 5 threes in 15 so the answer is 55."

In teaching arithmetic, we must start with the simplest, most meaningful method and work toward the mature algorism. For example, in the computation $37+25$, the steps would be as follows:

Step 1	*Step 2*		*Step 3*	*Step 4*
3 tens and 7 ones	30	7	37	37
+2 tens and 5 ones	+20	+ 5	+25	+25
5 tens and 12 ones	50	12	12	62
regrouping:	regrouping:		50	
6 tens and 2 ones or 62	62		62	

All the fundamental operations in arithmetic can and should be taught meaningfully about base ten in the beginning. When the child has a thorough understanding of what he is doing, then he is ready for the standard algorism (step 4)—the mature method of computation.

2. LEARNINGS SHOULD BE LOGICAL

There is a logical or common sense explanation for all arithmetical computations. In column addition we write ones under ones, tens under tens, etc.; and then we add ones, tens, and hundreds, carrying as necessary.

hundreds	tens	ones
		367
		495
		346

37
×24
——
148
72
——
868

In multiplying 37 by 24 the child should understand the logic involved. The 2 is written beneath the 4 since both digits represent tens, the 7 is written beneath the 1 since both represent hundreds. This is a far better explanation than instructing the child to "move over" when multiplying by the second figure.

The following amusing example used in figuring the cost of 7 tons of coal at $13 a ton shows a lack of understanding of the logic of our number system.

Method A		*Method B*	
$13		$13	
×7		13	(3 + 3 + 3
——		13	+ 3 + 3 +
21	(7×3=21)	13	3 + 3 + 1
7	(7×1=7)	13	+ 1 + 1 +
——		13	1 + 1 + 1
$28		13	+ 1 = 28)
		——	
		$28	

Checks:

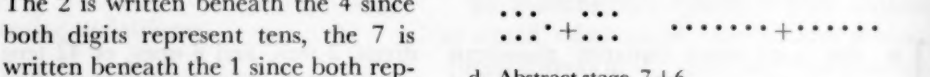
$13
7)$28 (8÷7=1)
7
——
21 (21÷7=3)
21
——

A child that comprehends the logic and structure of our number system will develop arithmetical meanings and insights that would never be possible by drill alone. Such a child frequently can proceed independently and solve computations with which he has had no previous experience.

3. LEARNING SHOULD PROGRESS GRADUALLY FROM THE CONCRETE TO THE ABSTRACT

With young children, abstract ideas in arithmetic develop as the result of many concrete learning experiences. For example, in teaching the basic combinations it is best to start with actual objects. In teaching a combination such as 7+6 the steps would be as follows:

a. Object stage, the child groups and adds 7 chairs and 6 chairs, 7 books and 6 books, 7 spools and 6 spools, etc.
b. Picture stage, the child adds pictures of 7 chairs and 6 chairs, 7 oranges and 6 oranges, 7 kittens and 6 kittens, etc.
c. Dot stage, the child adds dots and circles.



d. Abstract stage, 7+6

The same procedure can and should be used in teaching all the addition, subtraction, multiplication, and division facts as well as in the beginning stages of any new topic in arithmetic.

4. LEARNING IS ENHANCED BY PUPIL SELF-DISCOVERY

A child should be encouraged to discover the answers for himself. Such learnings are usually more meaningful and permanent. In the field of arithmetic, learnings by self-discovery are greatly fostered when the child has many concrete aids and devices at his disposal. Every classroom should have such simple devices as 10 wooden beads on a wire, 100 spools on a nail or peg board, a number pocket chart, a fraction board, etc. A spool board can be used to teach and learn rational counting to 100 by 1's, 2's, 5's, 10's; grouping and comparing; all the basic addition, subtraction, multiplication, and division facts; column addition; addition and subtraction with two-figure numbers; and partition. With 50 red, 25 green, 20 yellow, and 5 blue spools, the concepts of decimal and common fractions can be discovered. Concrete teaching aids and devices, besides being self-teaching, also add meaning and interest to arithmetic.

5. LEARNINGS SHOULD BE INTERRELATED

Studies have shown that some separate but related subjects or topics can be taught better together than separately. This principle of learning applies well to the teaching

of arithmetic. It is advisable to tie up and interrelate the four fundamental processes whenever possible. Teach addition and subtraction together; later multiplication and division should be tied in. When we teach $6+3$ and $3+6$, we should teach their reverses $9-3$, and $9-6$, as well as 3×3 and $9\div3$. Carrying in addition and "borrowing" in subtraction should be taught together since they are definitely related and an understanding of the concept of carrying readily leads to an understanding of borrowing or decomposition.

Whenever possible, fractions, both common and decimals, and percents should be taught together, since they are not only interrelated, but actually have the same values but expressed in different form. For example $\frac{1}{4}=.25=25/100=25\%=\$.25$.

The outmoded concept of teaching and drilling all processes in arithmetic separately certainly is not consistent with the findings of recent research. We now know that generalizations and meanings are much easier developed when we interrelate the processes and topics in arithmetic.

6. Learning Should Proceed from the Whole to the Part

Gestalt psychologists have discovered that better learning results when the child comprehends the whole before he begins work on the parts. This principle applied to arithmetic would mean that instead of teaching number facts as such, that we should start with the whole, say 8, and teach all its combinations (parts): $7+1=8$, $6+2=8$, $5+3=8$, $4+4=8$, $8-1=7$, $8-2=6$, $8-3=5$, $8-5=3$, $8-6=2$, $8-7=1$, $8\times1=8$, $4\times2=8$, $8\div2=4$, $8\div4=2$, and $8\div8=1$. After the child has learned all the parts of eight, select other numbers and repeat the process until all the basic facts have been mastered.

The number system could also be considered as constituting a larger but unified whole. Certainly a child never could grasp the basic meanings and insights involved in arithmetic unless he had an understanding of this whole and how it relates to its fundamental parts.

7. Drill Must Follow and Not Precede Learning

There are no investigations in the field of psychology that have shown that drill should be used as a method of teaching. However, studies would show that drill should be used as a means of maintaining skills and abilities which were originally taught by meaningful methods. No amount of drill will help a child progress to more mature methods of dealing with numbers. No amount of drill will teach the child the meanings of arithmetic. A child can be drilled 10,000 times that $4+3=8$, but when the child discovers the answer by meaningful techniques that $4+3=7$, he will give 7 as the answer and not 8, in spite of the 10,000 repetitions. A child may learn by sheer repetition that $7\times7=49$, but this does not insure that he knows that 49 is 4 tens and 9 ones, that 49 is 1 less than half a hundred, or could he necessarily use 7×7 to find the answer to 7×8.

A child is ready for drill in arithmetic only when he understands what he is doing and then drill serves its intended purpose—the retention and maintenance of skills.

8. Learnings Must Be Systematically Presented

The arithmetic program must be organized and sequential, since the simple processes must be mastered before the more difficult processes can be worked. A child who cannot add, cannot multiply when carrying is involved. A child is not ready for two-figure division until he can add, subtract, multiply, and do one-figure division. Analysis of errors' studies in arithmetic reveal that in almost any process, the major cause of errors is lack of ability in the subskills which the teacher has assumed that the child has mastered. We must be certain that before beginning a new process the child has the necessary background skills needed. One brick omitted from the wall of arithmetic will weaken and can topple the structure.

9. LEARNING SHOULD BE DIRECTED TOWARD FORESEEABLE GOALS

The purpose(s) and goals in learning must be understood by the learner, otherwise learning will be definitely hampered. Children must be able to see that they are learning to do arithmetic computations and to solve problems so that they can deal with arithmetical problems that arise in their other classes and in their out-of-school life. Furthermore, it is not difficult for a student to see the value of arithmetic if his vocational goal is in the area of science, mathematics, engineering, or business; in fact, almost any profession or occupation demands a certain degree of competence in the field of arithmetic.

10. THE LEARNER MUST BE MOTIVATED

A learner who is intrinsically motivated is likely to be a good student. Intrinsic motivation or motivation from within is readily obtainable with some children, but in many cases the teacher will need to motivate the child by other means. The use of number games, puzzles, riddles, and arithmetical tricks will add a great deal of motivation. In some cases it might be advisable to set aside a certain time each week for this. Following are several good examples of such motivating devices:

Write any number, multiply by 2, add 18, divide by 2, subtract the original number. The answer will always be 9.

Write your age, multiply by 2, add 5, multiply by 50, subtract the days in the year (365), add the change in your pocket under one dollar, add 115. The first two numbers of the answer will be your age, the other two numbers in the answer will be the amount of change in your pocket.

Complete the following multiplications:

$1\times8+1=9$	$1\times9+2=11$
$12\times8+2=98$	$12\times9+3=111$
$123\times8+3=987$	$123\times9+4=1111$

In recent years an increasing emphasis has been placed on the development of more and better mathematicians. However, we must not lose sight of the fact that a thorough understanding of arithmetic is essential to later success in mathematics. By applying the proven principles of learning to the teaching of arithmetic, we will not only improve the teaching of arithmetic, but also lay a solid foundation for those who are later to excel in mathematics.

EDITOR'S NOTE. Within the realm of accepted psychological principles there remain large areas of judgment as to sequence and procedure of learning. How soon should one proceed from a multiple-step procedure which is used in the early stages for the development of understanding to the final more economical process? How much visual and manipulative material should be employed and when should the work proceed to the more abstract and symbolic stage? What to do with youngsters who grasp the meaning and significance of a principle quickly while their colleagues are floundering?

Many people will doubt Dr. Koenker's wisdom in suggesting that we learn all about 8 in terms of addition, subtraction, multiplication, and division combinations. They believe that it is more valuable to a child to fix a process, as for example addition, more firmly before thinking of division. The combination of simple addition and subtraction facts seems a sensible procedure at certain stages. On the other hand, it might be better to concentrate on carrying in addition before decomposition in subtraction is attempted. True, the one is useful in explaining the other. One of the functions of drill is to "fix" a response so that a pupil can answer almost automatically. It has been the editor's experience that it is sometimes rather difficult to unlearn an incorrect response once it is firmly fixed in mind.

"She told you 5 and 2 make 7? THAT does it! She told me 4 and 3."

Reprinted from The Chicago Tribune

Ideas for Your "Bag of Tricks"

HUMPHREY C. JACKSON
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THE EXPERIENCED TEACHER has a *bag of tricks* of the trade at her command every day and uses a wide variety of them in the daily work of the classroom. You have doubtless heard a great deal recently about the *discovery technique* because in the hands of a skilled teacher it is said to provide (1) interest for the learner, and (2) satisfaction to the teacher. In our modern world, methods used a generation ago do not seem to create the same interest on the part of the student. The student of today needs to feel that today's methods are the newest, best, and most effective. With the advent of television, scientific progress, electronic computing machines, space satellites, supersonic speeds, intercontinental missiles and the like, he has a right to expect teachers who are alert, well informed, specialists in their fields and who can employ effective, modern methods of instruction.

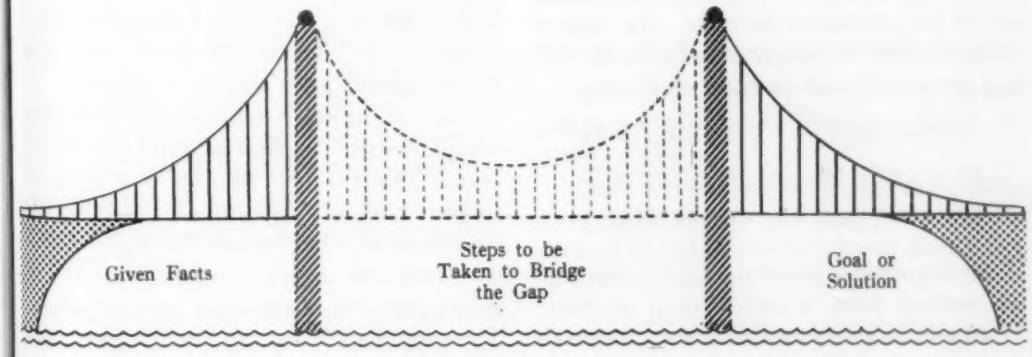
Building Bridges

One teacher in a junior high school arithmetic class uses the idea of "building bridges" when teaching problem solving. Here the given facts are compared to one end of the bridge, and the goal or solution to be reached is considered as the other end of the bridge. The center span is compared to the steps necessary to bridge this span.

When building a bridge, often there are a series of things to be done and which must be done in a planned order in order to complete the structure properly. The same order of steps is often necessary when solving a problem in mathematics. The student is frequently able to solve problems by applying only one or two steps between the given facts and the final solution. Gradually this leads to problems of greater complexity and a third and fourth step is found necessary in the more difficult problems. By the time the student reaches problem solving in Algebra, he is able to solve equations which require many steps before the solution is reached.

Factories Make Products

Many years ago teachers of mathematics discarded the "Type I, Type II, Type III" approach to the teaching of percentage. Modern thinking leads one to the recognition of the *ratio* approach inasmuch as the formula for per cent $r \times b = p$ is one of many formulas of this type commonly used in the solution of problems. For example, $n \times p = c$,



the formula for finding the cost of a number of articles selling at a unit price, is one which is found early in many texts.

The new approach to this type of problem involves a recognition on the part of the student of *factors* and *products*. Therefore, introductory lessons are given in the recognition of these terms. Examples are taken from arithmetic such as the multiplication tables. If $3\times4=12$, which numbers are factors and which product?

Those familiar with the Illinois Study now being conducted under the direction of Dr. Max Beberman will recognize here a wonderful opportunity to use his technique in the preparatory work leading to the recognition of factors and products.

The teacher may start by writing on the board a problem like $3\times5=\square$. The class is then asked for the correct number which should be written in the square. Which numbers are factors and which number is the product? $\square\times\square=24$.

Several other problems of this type are written on the board with similar questions about each. The teacher might seek for the observation that when both factors are known, you multiply to find the product.

One day, after presenting several problems on the recognition of factors and products, a teacher asked the students to name the given factor in the problem $3\times\square=21$. One pupil gave the correct response and was asked how he knew 3 was the given factor. His reply was simply that he thought of factories and then that *factories make products.*

Variation in the unknown can be used in the examples suggested by the teacher. Examples can be varied by placing the product on the left instead of the right. The student should be able to recognize and name factors and products in any position suggested.

(1) $4\times5=\square$, (2) $3\times\square=24$,
(3) $\square\times6=18$, (4) $\square=5\times7$,
(5) $42=\square\times7$, (6) $39=3\times\square$.

Some questions asked should be, what are we seeking here, a factor or a product? What are the given numbers, factors or products, name each? How do you find the missing number, by multiplication or division? When do you use multiplication? When do you use division? When you use division, is it the ratio between the product and the known factor?

It should be only a step from the initial practice lessons to working with letters and formulas. One might suggest the following $f\times F=p$ as a possible formula for this type of problem, where small f represents one factor, capital F represents a second factor and the letter p represents the product.

Work with the letters, expressing how each letter might be found by indicating the method (multiplication or division) of computation expressed between the other two letters. The student will become aware that when solving for "p" he should multiply the two factors; when solving for one of the factors, he should divide the product by the other factor.

Now use the formula $r\times b=p$, where r represents the rate of per cent, b the base and p the percentage (part). It should be easy to establish the relationship between r and b when solving for p; between p and r when solving for b; and between p and b when solving for r. Thus the student will become aware of how to solve any problem involving the factors and product without ever having a suggestion of Type I, Type II, or Type III.

Generalizations can easily be drawn by the class in summarizing their observations. One such generalization is that the rate of per cent is the ratio between two numbers represented in the formula by p and b. Thus we have accomplished the teaching of per cent as a ratio between two numbers.

It might be pointed out that the Commission on Mathematics of the College Entrance Examination Board advocates the *ratio concept approach to the teaching of per cent.*

Some of the newer texts now available are complete with Teachers Manual and Workbooks. The Teachers Manual is a veritable treasure chest of suggestions for assisting the teacher to plan her lessons. Particular attention and care is given in these manuals to techniques for introducing new concepts. These manuals put at the teachers finger tips enrichment materials available either by

renting or lending and in many instances explains how to make such materials. The do-it-yourself technique is advocated and suggestions for low-cost items make such projects much easier to carry out.

Is your school considering adopting a new arithmetic textbook soon? If so, be sure to inspect the newest books, some of which are just off the press, copyright 1958. Check the books you examine against criteria such as the College Examination Board[1] recommendations to make sure they meet the requirements of good established practice. You want your students to have the best modern texts available. Have your textbook selection committee study available books and develop a checklist of criteria to help them select the best text for their needs.

[1] *The Mathematics of the seventh and eighth grades,* A program recommended by the Commission on Mathematics of the College Entrance Examination Board, copyright 1957. Available by writing the Executive Director, Commission on Mathematics, College Entrance Examination Board, 425 West 117 Street, New York 27, New York.

EDITOR'S NOTE. Every good teacher has developed a number of "tricks of the trade" which are used when opportune to develop interest, to foster a discovery, or to challenge pupils to think and to learn. It is a good idea to share some of these as Mr. Jackson is doing. But, the "trick" of one teacher may not serve equally well for another. Of course, these are not really tricks in the common sense, they are in fact devices and procedures that have valid educational application. Who else has some good procedures that should be shared with others?

Divisibility by Seven and Thirteen

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IN HIS ARTICLE *Divisibility and Prime Numbers* in the March 1958 issue of THE ARITHMETIC TEACHER, Mr. Paul Yearout states: "The only number less than twelve for which there is no simple rule [of divisibility] is 7. For seven (and also thirteen) there are rules, but they are complicated and take almost as much time as long division."

Exactly which rules Mr. Yearout had reference to with respect to the divisors of 7 and 13, he did not state. However, since rules of any kind for divisibility by these two numbers are rarely found in such treatments, it seems advisable, in this instance at least, to complete the picture with a pair of rather simple rules for ascertaining exact divisibility by either 7 or 13.

TEST FOR DIVISIBILITY BY SEVEN: *A whole number is exactly divisible by 7 if the difference between twice its units-digit and the number formed by its non-unit digits is exactly divisible by 7.*

Example 1. Is 294 exactly divisible by 7?

For 294, 4 is the units-digit and 29 would be the number formed by the non-unit digits of 294. Then, twice 4 is 8; $29-8=21$; 21 is exactly divisible by 7. Consequently, 294 is exactly divisible by 7.

Example 2. Is 436 exactly divisible by 7?

Twice 6 is 12; $43-12=31$; 31 is not exactly divisible by 7. Therefore, 436 is *not* exactly divisible by 7.

When the number in question contains many digits, the test may be repeated until a result is obtained in which exact divisibility by 7, or the lack of it, is obvious.

Example 3. Is 30,275 exactly divisible by 7?

In 30,275, the units-digit is 5 and 3.027 is

the number formed by the non-unit digits. Twice 5 is 10; $3,027-10=3.017$.

Since 30,275 is exactly divisible by 7 if 3,017 is, the test may be repeated to determine whether 3,017 is exactly divisible by 7: the number formed by the non-unit digits (301) of 3,017 when diminished by twice its units-digit (14) is 287.

If the fact that 287 is exactly divisible by 7 is not immediately apparent, then the test may be repeated a third time:

$$287 \rightarrow 28-(2\times 7)=14$$

Since 14 is exactly divisible by 7, then 287 is; and since 287 is exactly divisible by 7, then 3,017 is; and since 3,017 is exactly divisible by 7, then 30,275 is.

TEST FOR DIVISIBILITY BY THIRTEEN: *A whole number is exactly divisible by 13 if the sum of four times the units-digit added to the number formed by its non-unit digits is exactly divisible by 13.*

Example 1. Is 117 exactly divisible by 13?

For 117, the units-digit is 7 and the number formed by the non-unit digits of 117 would be 11. Then, four times 7 is 28; $11+28=39$; 39 is exactly divisible by 13. Consequently, 117 is exactly divisible by 13.

Example 2. Is 325 exactly divisible by 13?

Four times 5 is 20; $32+20=52$, which is exactly divisible by 13. If this latter fact about 52 is not obvious, the test may be repeated: four times 2 is 8; $5+8=13$, and 13 is clearly divisible by 13. Therefore, 325 is exactly divisible by 13.

Example 3. Is 42,317 exactly divisible by 13?

$$42,317 \rightarrow 4\times 7=28;\ 4,231+28=4,259$$

$$4,259 \rightarrow 4\times 9=36;\quad 425+36=461$$

$$461 \rightarrow 4\times 1=4;\quad 46+4=50$$

$$50 \rightarrow 4\times 0=0;\quad 5+0=5$$

Since 5 is not exactly divisible by 13, then neither is 50, nor 461, nor 4,259, nor 42,317 —the number originally in question.

Proofs of Divisibility

For those who are equipped to follow the algebraic reasoning by which the generality of these two rules may be established, the following is offered:

Divisibility by 7

Any multi-digit whole number (N) may be expressed as $10t+u$. Let R be the remainder after twice the units-digit ($2u$) has been subtracted from the number formed by the non-unit digits (t). Then

$$N=10t+u$$
$$u=N-10t$$
$$2u=2N-20t$$
$$R=t-2u$$
$$=t-(2N-20t)$$
$$=t-2N+20t$$
$$=21t-2N$$

Since 7 will always divide the $21t$ of R exactly because of the factor 21, it follows that R will have an integral quotient when divided by 7 *only* when 7 divides N exactly.

Divisibility by 13

Any multi-digit whole number (N) may be expressed as $10t+u$. Let S be the sum after four times the units-digit ($4u$) has been added to the number formed by the non-unit digits (t). Then

$$N=10t+u$$
$$u=N-10t$$
$$4u=4N-40t$$
$$S=t+4u$$
$$=t+(4N-40t)$$
$$=t+4N-40t$$
$$=4n-39t$$

Since 13 will always divide the $39t$ of S exactly, regardless of the integral value of t, then S will have an integral quotient when divided by 13 *only* when N is exactly divisible by 13.

Quantitative Thinking in Today's World

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TWENTIETH CENTURY MAN, living in a constantly changing scientific and technological world, is called upon every day to form judgments, make decisions, and take actions concerning problems relative to some quantitative aspect of his environment.

Some people present the argument that much of the mathematical work of today is done by machines, and it must be admitted that scientists and mathematicians have developed phenomenal machines which can, in a few minutes, complete the mathematical computations that would take many expert mathematicians several hours to accomplish. That man is still the thinking element which is necessary to the solution of mathematical problems is a requisite factor which should not be forgotten. Even though machines can perform the computational aspect more efficiently than can man without the aid of such machines, the fact still remains that the machine's solution to a mathematical problem is no better than the quantitative thinking or reasoning of the person who developed the machine and of the person who "feeds" the facts into the machine. Machines may well be used to do many of our calculations but human beings must interpret the results. Thus, few people would deny that a real need exists in present-day society for functional mathematical understandings and effective quantitative thinking.

It should be emphasized that the development of computational ability is and will undoubtedly continue to be an important aspect of mathematical instruction at the elementary-school level. The author is saying, however, that computational ability is not enough! Understanding and reasoning are just as important, if not more important, than computational skill. It is imperative that children see sense in what they learn.

Teachers should make every attempt to see that pupils gain insight into the various arithmetical concepts and processes. The ability to perform various processes and operations may be acquired through repetitive drill, but unless the pupil understands what is involved in the processes and operations and is aware of relationships existing between them, he will be unable to use judiciously his arithmetical knowledge. The ability to think quantitatively cannot be developed through mere memorization or meaningless drill. When properly taught, arithmetic makes a real contribution to the development of sound reasoning and clear thinking.

Quantitative thinking is constantly required in carrying forward normal everyday enterprises both in and out of school. There is no end to the situations involving the need for mathematical interpretation. If children are to lead economically and socially competent lives, they must be helped to understand number as such, and to see its application to daily life.

If arithmetic is to facilitate reasoning in quantitative situations, the instructional program must provide children more opportunities to use their higher mental processes. The instructional program should be such that pupils soon discover that school arithmetic involves much more than the ability to "juggle figures." Arithmetic experiences should call for judgment and action on the part of the individual learner.

The understandings, meanings, and reasoning ability developed in the elementary school arithmetic program constitute the basis for all future work in mathematics. It is extremely important, therefore, that by the end of the sixth grade children will have developed a fair degree of competency in

sensing, understanding, and responding to the quantitative aspects of daily life.

During the early part of this century emphasis in the teaching of arithmetic was on mastery of skills through drill. The emphasis today is on the development of meaningful concepts and quantitative thinking. If arithmetic is to contribute to intelligent and enriched living, teachers must help children to see that arithmetic is useful and that it makes sense. Schools have a definite obligation to help the learner develop: (1) understanding and proficiency in the various number operations, (2) understanding of the number system and meaningful number concepts, and (3) quantitative thinking—the ability to apply number, and mathematical concepts and processes intelligently and skillfully to the quantitative situations encountered both inside and outside the classroom.

Zero and Infinity

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ZERO, FOR MOST YOUNGSTERS, has but one meaning. To them zero means "nothing." Phrases such as "five to nothing" heard at the ball park to indicate a score of 5 runs to 0 runs helps to substantiate this common belief. In subtracting six apples from six apples, one is naturally led to believe that the answer zero means that nothing is left. Although this reasoning is correct in as far as we have gone, it is not entirely without fault. A student entering junior high school must learn that the concept of zero has a broader meaning. If a student does not learn to accept a broad point of view, he will experience great difficulty in future mathematics courses.

Let us first examine the following definition: Zero is a number which exhibits the following properties:
For any number x:

$$0+x=x+0=x$$
$$0\times x=0$$

Note that zero is now classified as a number, but one which has properties unlike other numbers. Notice that if zero means nothing, a pupil may be led to reason that when nothing is multiplied by x, the following conclusion is necessary: $0\times x=x$, for the x is bound to remain unchanged if nothing is done to it!

Again you might be fooled by the use of zero in grading. A student's grade in a mathematics course is nothing if he never took the course. On the other hand, if he did take the course and his work was judged worthless, his grade would be zero.

In the elementary grades students learned that good mathematics requires a check, and so multiplication was used to check a division problem. For example: $\frac{6}{2}=3$. Checking, we multiply 3×2 and get 6. Since this number is the same as the numerator (dividend) in the original division problem, we say that the answer is correct. Using zero we have $\frac{6}{0}=$ an unknown number. Since the check gives us zero times any number will equal zero, we must conclude that division by zero is impossible.

However, when we graph equations such as $Y=\tan X$ we obtain a curve that approaches closer and closer to a line parallel to the Y axis as the values of X increase toward 90°. Similarly our number system beginning at a zero point may extend indefinitely in either direction along a line. Likewise a constant such as the number 6, when divided by a decreasing value approaching zero produces a quotient which is increasing indefinitely.

$$\frac{6}{3}=2;\ \frac{6}{2}=3;\ \frac{6}{1}=6;$$
$$\frac{6}{\frac{1}{2}}=12;\ \frac{6}{1/1000}=6000;\ \text{etc.}$$

We say that the answer is approaching *infinity* for which we use the symbol ∞.

Numbers are the invention of man. Just as zero was created to fill a need so is the concept of infinity. But neither has value unless we learn to understand and to use it in its proper significance.

Do They See the Point?

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DO THEY See the Point? This question has a dual meaning. "The Point" refers both to the actual dot, i.e. the decimal point, and to the "point" or meaning of the expression wherein the decimal point is used. Likewise two forms of seeing are implied. The one has to do with the adequacy of the sense of *sight*. The other is concerned with ideological understanding and judgments of significance, i.e., *insight*. The one refers to visual sensitivity. The other to conceptual sensibility. Both are important and worthy of careful consideration.

The visibility of the decimal point in mathematical printing or writing is often given little regard. The symbol commonly corresponds in size with the dot of the *i* or the period dot used. However, both the dot and the period generally have configurational cues within the structure of the word or sentence which make them likely to be observed.

In most decimal number expressions there is little of configurational cue as to the location of a decimal point. Its location is determined by the quantitative amount which the number is designed to express. The decimal point serves to "point out," i.e. to locate the units' place within the number expression. Consequently the decimal point needs to be as "readable" as the other aspects of the number symbol. Failure to see a decimal point or failure to "see" the implication of a decimal point can be a source of serious failure in communication.

An analysis of the incorrect responses produced by several hundreds of children in grades 7 and 8 of a city school discloses evidence of possible failure to see the decimal point. Some of these failures may well have been *sight* failures. The others are more properly classed as deficiencies with *vision*.

The omission of a necessary decimal point in the answer was one type of characteristic error. We have no way of determining whether the omissions were due mainly to *sight* deficiencies or to those of vision, i.e. intellectual deficiencies, but that sight may have been a significant factor is suggested by the following data:

(1) When a decimal point occurred in the example in the dividend only, the decimal point was omitted in twenty-two per cent of the incorrect answers;
(2) When no decimal point occurred in either the divisor or dividend, a necessary decimal point was omitted in fifty-seven per cent of the incorrect answers;
(3) When a decimal point occurred in both the dividend and divisor, the decimal point was omitted in only eight per cent of the incorrect answers; and,
(4) When a decimal point occurred in the divisor only, the decimal point was omitted in twenty-three per cent of the incorrect answers.

The omission of a decimal point where one is needed is, demonstrably, an important source of error. The wide variance between the proportion of omissions when no decimal points occurred in the statement of the example, fifty-seven per cent, and the proportion of omissions when decimal points occurred in both the dividends and the divisors, eight per cent, suggests that the visual sensing of the decimal point may tend to provoke one to use a decimal point when he might not otherwise do so. Hence, we recommend that serious attention be given to make the decimal points visually legible. As a purely arbitrary standard until a better basis for legibility can be determined, we recommend that the decimal point have double the diameter of the period or of the dot of the *i* used in printing or writing. Readers need to *see* the decimal point.

Visibility is a *prerequisite to* but it certainly is no *guarantee of* "seeing the point" in the intellectual sense of that expression. One may see the point in the sight awareness sense, but not "see the point" of it as a symbol of quantitative significance. Failure to see the point as a sign of quantitative value is a perceptual rather than a receptual deficiency.

If placing a decimal point in the answer can be accepted as evidence that the pupil had awareness of it, then misplacing the decimal point may comparably be interpreted as a failure to understand its meaning or significance. Let us examine how the children mentioned previously performed in this regard. The decimal points which they placed in answers to examples with decimal points in their dividends only were misplaced in thirty per cent of the incorrect answers. The decimal points were misplaced in twenty-one per cent of the incorrect answers when no decimal points occurred in either the dividend or the divisor.

Misplaced decimal points occurred in forty-two per cent of the incorrect answers when decimal points appeared in both the dividend and divisor of the examples. And, decimal points were misplaced in fifty-nine per cent of the incorrect answers when a decimal point occurred only in the divisor of the example.

The discrepancies of omitting the decimal point or misplacing it occurred in roughly fifty-two per cent of the incorrect answers when a decimal point occurred in the dividend only; in seventy-eight per cent of the incorrect answers when no decimal point appeared in the example as given; in fifty per cent of the incorrect answers when a decimal point occurred in both the divisor and the dividend, and in eighty-two per cent of the incorrect answers when a decimal point was presented in only the divisor of the example. Failure on the part of the pupils to see the point merits the attention of teachers.

Compared with the errors with the decimal point errors the number aspects of division were much less numerous. With examples having a decimal point in the dividend only, approximately eighteen per cent of the incorrect answers contained a number fact error. With examples having no decimal point, nine per cent of the incorrect answers contained a number fact error. Examples with decimal points occurring in both the divisor and dividend produced twenty-five per cent of incorrect answers contained in the number facts. Examples with a decimal point in the divisor only produced answers of which fifty-five per cent were incorrect but only nine per cent of the incorrect answers contained a number fact error. As reported above, eighty-two per cent of the incorrect answers contained a decimal point error. Approximately nine per cent of the "incorrect" responses were instances of omission or failure to attempt a solution.

Summary. This paper attacks the problem of "seeing the point" as regards the decimal point from two points of view, viz., visual sensing and understanding. Omissions of decimal points were classed as possible sight deficiencies. Since this occurred with considerable frequency, it is proposed that care be taken to make decimal points clearly legible.

Misplacement of decimal points were classed as failures to "see the point" of the decimal symbol intellectually. More attention needs to be given to the development of meaning and judgments of significance. When a pupil writes a decimal point, he should be expressing an idea rather than performing a computational skill.

EDITOR'S NOTE. Professor Spencer has called attention to the factors of "sight" and "insight" connected with the use of the decimal point. Since we seem to have standardized upon the use of a "dot-point" in this country and we place it at the base line with a row of digits it would seem very wise to adopt the Spencer suggestion of printing the dot larger than we normally do. Yes, let us create a new type symbol to use for the decimal point. The dot is acceptable but it should be clearly visible. Professor Spencer suggests a dot twice the diameter of the dot above the letter *i*. The problem of insight is more involved and requires some honest attention in teaching and learning. Here is another area where it is desirable to teach by a method that promotes understanding and aids and requires pupils to think.

Arithmetic in Verse

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"LET'S MAKE MATHEMATICS EASY!" is often times given as the solution to the difficulties experienced in acquiring this discipline. We need only pick up a Sunday newspaper to read an advertisement for a simplified method of learning some phase of mathematics. Back in the 1840's a clever, enterprising man named George Van Waters felt he had found a solution to the difficulties in acquiring arithmetic.

Van Waters may have reasoned in this way. If only we could reduce those difficult steps in computation to a verse or phrase that could be easily remembered. There is no difficulty in remembering the familiar, "Thirty days hath September . . .," because it is written in a rhyming form. Van Waters felt he had found a key to easier learning. He started writing verses.

Van Waters published his original *Poetical Geography* in 1841 and later in 1847 added arithmetic under the title *Poetical Geography with Rules of Arithmetic in Verse*. Many newspapers of the day accorded acclaim to the author for his ingenuity. Readers may be interested in some of the rules which are given below.

Although this verse method received acclaim in Van Water's day, it is easy to see why it did not succeed. By putting the learning of arithmetic in verse form, he apparently did not regard this subject as a well-organized body of knowledge that depends upon basic mathematical laws, but rather as merely a series of manipulations and memorizations for arriving at an answer.

In contrast to this attitude is the work of Warren Colburn, who published *Intellectual Arithmetic upon the Inductive Method of Instruction*. Colburn, a contemporary of Van Waters, placed emphasis upon understanding of the arithmetical processes. His method was to introduce a process by practical examples in which the pupil would experiment in his own way. After this the process would be taught and more abstract examples considered.[1]

Lyman Cobb, another textbook writer of that day, varied his approach considerably from Colburn, but his general aim was also for comprehension of the process. He would present the rule of arithmetic first, then proceed with an explanation of this rule, followed by examples.[2]

It should be pointed out that others before Van Waters had placed the rules of arithmetic in verse. The anonymous *Marmaduke Multiply*, first published in England during 1816–17 and followed by several American editions, has a wood-cut on each page and an arithmetical rhyme. But earlier than this in 1600, Thomas Hylles published *The Arte of Vulgar Arithmetic*, which also placed the rules in rhyme. Hylles felt his idea was original as he claimed that, "The whole method and metrizing thereof is mine own and such as no man hitherto hath used, either in English or Latine to my knowledge." Hylles was mistaken, as Dionis Gray had already used this method in 1577.[3]

In comparing these earlier verses with those of Van Waters, it might be said that probably no one has placed the rules of arithmetic in verse so extensively and completely as did George Van Waters.

[1] Warren Colburn, *Intellectual Arithmetic upon the Inductive Method of Instruction*, Hilliard, Gray, and Co., Boston, Mass., 1839, pp. 3–9.

[2] Lyman Cobb, *Explanatory Arithmetick*, Harper & Bros. New York, 1834, pp. iv–vi.

[3] Florence A. Yeldham, *The Teaching of Arithmetic Through Four Hundred Years* (1535–1935), George C. Harrop & Co. Ltd., London, 1936, p. 52.

THE

RULES OF ARITHMETIC

IN VERSE.

Addition.

Addition, is joining more numbers than one,
And putting together to make a whole sum,
Addition's the rule that learns us to count,
And the sum that 's produced is called the *amount*.

RULE.

The numbers write down, as the rule comprehends,
Placing units under units, and tens under tens;
Draw a line underneath, and commence at the right,
Or the unit column, the work to unite;
If its sum or amount should not exceed 9,
Then place it direct 'neath its own native line:
But if 9 it exceeds, then the unit you place
'Neath the column of units, (the units to grace);
While the tens or the figure that's to the left hand,
To the next column join, as you well understand.
Observe the same rule, till you come to the last,
And the whole amount write as this column you cast.

Subtraction.

Subtraction, it teaches, when numbers are given,
One greater, one less, as 10 stands to 7,
To find out their difference, for difference we see,
And when worked and achieved, we find to be 3.

RULE.

The numbers first write, the less under the greater,
Placing units and tens, in lines of their nature,—
The subtrahend, then, from the minuend take,
And that, which remains, an answer will make.—
But if in the less number, a figure we find,
Which exceeds that above it, let 10 then be joined
To the figure above, and from the amount,
Take the figure below, (nor mistake in the count),
But forget not to add, to the next figure, then
In the subtrahend, *one* to make up for this *ten*.

Addition.

ADDITION is joining together two or more numbers, to make one whole sum or amount.

Addition is the rule by which we count, or put numbers together.

The whole sum, or answer, is called the *amount*.

RULE.

Write down the numbers, one under the other, placing units under units, tens under tens, and draw a line underneath.

Begin at the right hand, or unit column, to add or unite the numbers together; add together all the figures contained in that column.

If the sum or amount should not exceed 9, then place it under the column; but if it does exceed 9, put the right hand figure under the column, and carry the left hand figure, and add it on to the next column.

Observe the same rule, putting down under the column added, the right hand figure, if it exceeds 9; and carrying the left hand figure to the next column.

At the last column, write down the whole amount, and the work is complete.

Subtraction.

SUBTRACTION is taking a less number from a greater to find out the difference, as 7 from 10; the difference, or remainder, is 3.

The greater number, or the number to be lessened, is called the *minuend*. The less number, or the one to be taken from the greater, is called the *subtrahend*. The difference, or that which is left after the operation of the work, is called the *remainder*.

RULE.

Write down the numbers, the less under the greater, placing units under units, tens under tens, and draw a line underneath.

Subtract the less from the greater: commence at the right hand figure in the lower line, and take it from the one above it in the upper line; write the difference below the line. So proceed till the whole is subtracted.

If the figure above should be less than the one below, then add *ten* to the one above, and from the amount, take the figure below. But in this case you must add one to the next left hand figure, in the lower column. This is called borrowing ten.

Perspective in Arithmetic*

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MANY YEARS AGO I read Creasy's "Fifteen Decisive Battles of the World." These were thrilling to me. Before each battle, civilization was pictured as at a crossroads. A momentous issue was to be decided once and for all time.

In the first of these battles, the matured despotism of Asia was pitted against the budding democracy of Greece We see Athenian generals on a slope overlooking the plain of Marathon. Creasy says that on their deliberations "depended not only the fate of two armies but the whole future progress of human civilization."

In almost every field of human endeavor, the milestones of advance are matters of record, to guide and to inspire.

Reference books give long lists of peacetime discoveries and inventions that mark turning points in world history. The first use of iron; the use of moveable type in printing; the steam engine! Life has never been the same after such a discovery or invention as it had been before. There is no return to oil lamps or candles after electric lights come into general use. Battles won in science and technology have proved as decisive as battles won by armies.

Permanent advances have been made and are being made in medicine, surgery, hospital care. Scientific medicine respects the work of Lister, Pasteur, Koch. Such work is not accepted as final, but as sound—something upon which later workers can build with confidence. The support given to medical education and medical research is evidence that the public respects scientific medicine. The medical researcher wins battles that stay won.

Records of the development of mathematics as such—histories of mathematics, biographies of great mathematicians, source books of the advances that have made mathematical history—are in any college library.

Now, consider arithmetic and especially the learning theories that apply to arithmetic.

Arithmetic has been a common school subject in this country for two hundred years. It has been a part of man's culture for many thousand years. Yet, we have not one authentic, comprehensive record of decisive battles. No hard core of steady advance has been identified for us. We move this way and that, deprived of the inspiration and guidance that come from knowing that we build upon the solid work of great teachers of the past.

Hundreds of intelligent, conscientious men and women are working—giving the best they have—right now to build better learning programs in arithmetic. Such was true fifty years ago, a hundred years ago, two hundred years ago. Neither now, nor then, has the worker for a better arithmetic had the help that he should from the past. Articulate ones in each era have chosen to picture the past of school arithmetic as wasteland and void. The emphasis has been on a little newness, with a de-emphasis on the big, solid base of content and method inherited from long lines of great teachers.

To illustrate emphasis of new and de-emphasis of old:

A recent issue of *Saturday Review* describes current attempts to improve a mathematics learning program: " . . . an entirely new method . . . every teacher must be especially trained."

A recent issue of *Educational Leadership*

* This paper was presented at the meeting of the National Council of Teachers of Mathematics at Cleveland, April 12, 1958.

says: "The training they (teachers) received even a decade ago is inadequate today either as to substance or as to methodology."

But even this—the invidious comparing of the old with the new—is itself old. In *The New Methods in Arithmetic*, Thorndike, 1921, tells again and again the advantages of his "newer" methods as compared with the "older" methods of others. Charles Davies in his book on teaching arithmetic, 1850, lauds his "new system of instruction." Warren Colburn, 1830, describes the bad situation that existed "prior to the introduction of the new system."

An overemphasis on the bit that may be new, unmindful of the great contribution of the past hurts in two ways.

First, the generals—those in university or research positions, with opportunity and obligation to work toward better learning programs, are dissipating their energies fighting the same old battles over and over again. No battles stay won. It is difficulty to identify any hard core of a steady advance.

Second, practical school administrators and room teachers are slow in accepting any change. They sense a lack of steadiness on the part of the generals. And as St. Paul said, " . . . if the trumpet give an uncertain sound, who shall prepare himself for battle?"

I offer the following as constructive suggestions.

We need an authoritative, comprehensive, readable history of the teaching of arithmetic. We have nothing in English that even approximates what I have in mind. This history should include reprints of the classics in the development of arithmetic. It should include summaries of all significant philosophies and researches that have influenced the development of arithmetic. It should include short biographies of the men who made the philosophies, the researches, the texts, and the courses of study in arithmetic. It should include a bibliography of all relevant material. To serve its purpose this history must extend over centuries rather than decades. It must not be a mere "catch all." Scholarship, insight, intellectual integrity must go into the selection and organization of materials, that the history show decisive battles and hard core of advance. If there have been no decisive battles and there is no hard core of advance, let that be shown.

The materials of such a history are scattered and not easily accessible. The past is not serving us as inspiration or as guide. We are confused by conflicting theories and by petty issues magnified into great ones. There is no authoritative source to which we can go for an answer.

We have a few histories of arithmetic, as those by Cajori and Karpinski. These are old. Their emphasis is on arithmetic content. Little attention is given to underlying philosophies and theories of learning. No attempt is made to spotlight decisive battles, or to point out any hard core in a steady advance.

We have the arithmetic bulletins published by the University of Indiana fifteen years ago. These describe arithmetic texts used in this country. They make no attempt to identify a core of advance. No decisive battles are identified. Much milling around is in evidence as one compares the different arithmetics.

We have a thousand prefaces of arithmetics published in the last two hundred years. These highlight the arithmetic content and learning theories of the several authors. . . . What stands out most in these prefaces? Is it a hard core of steady advance, or is it a see-sawing back and forth? These prefaces should be studied critically to untangle the advances from the see-sawing.

For the past two hundred years, in our country, theories of learning have been described in treatises on the art and science of teaching arithmetic, in speeches and articles in journals and yearbooks, in courses of study and special bulletins. How much of this has built toward a hard core in a steady advance, and how much has been spent in fighting windmills and creating puffballs? What decisive battles are recorded in this material? We cannot answer these questions until a critical study has separated

the passing and petty from the permanent and potent.

Our nation grew to greatness while studying arithmetics written by Pike, Daniel Adams, Daboll, the two Colburns, Emerson, Davies, Greenleaf, Thompson, Ray, Feltner, White, Milne, Stone, ... Any school child can read for weeks about the life, works, hobbies, and idiosyncrasies of Longfellow, Johnny Appleseed, or Buffalo Bill. Where could the child look to find anything about Greenleaf, Ray, or Milne? Such men were knowing enough to lay the arithmetic foundation for a great nation. Now, they and their works are non-significant in a wasteland and void of the past of arithmetic.

The implications of what I say include:

First, we take no pride in arithmetic's past.

Second, we have set up no stable list of decisive battles.

Third, we identify no hard core in a steady advance.

An authoritative, comprehensive, readable history of the teaching of arithmetic could help us get on the track and stay on the track in a steady advance. Having a record of the past does not carry with it any idea of returning to the arithmetic of the past, any more than we are now doing. Such a record should help us see arithmetic growing rather than arithmetic stagnant. In such a record, a thousand threads of arithmetic learning could be traced through the years. I mention in barest outline one such thread: Learning arithmetic by discovering the basic principles for oneself.

Margaret Drummond (1925) wrote: "Children . . . should not be taught what they can find out for themselves." She has a pupil say: If I learn to discover little things when I am little, I will be able to discover big things when I am big.

The preface to the Cook-Cropsey arithmetic (1896) describes the work as "designed to throw the pupil on his own resources."

Laisant (about 1890) said: "The problem is always the same: to interest the pupil, to induce research, to give him the notion continually, the illusion, if you please, that he is discovering for himself that which is being taught him."

The preface of Emerson's arithmetic (1834) encourages the "habit of investigating," and "discovering the true relations."

On the other side of the ledger are many statements of the value of being able to understand and to follow directions, and to learn from others. Life is too short to learn everything by personal discovery. Many who would never discover a principle for themselves are able to understand and use the principle.

Seventy years ago, Simon Newcomb said: "Poring over problems vainly is time wasted. . . . Explain the problem when the pupil does not readily see it."

Forty years ago, David Eugene Smith said: "No extreme of 'method' should be adopted . . . to get into any narrow rut whatever . . . is to fail of the best teaching and to narrow the horizon of the children in our care."

Woodrow Wilson wrote of thinking everything out for himself: "I not only use all the brains I have, but all that I can borrow."

There is not a decade but that the discovery method is offered as a new wonderway to cure all the ills in arithmetic learning programs. Fervor rises and falls. There is one un-recorded skirmish after another. There is no record of a decisive battle—no record of the contributions of the discovery method in a steady advance.

Through the years teachers have been blinded by enthusiasts of one narrow way after another. She should see all important methods and minor variations in proper perspective, in order to be a good teacher. Material should be made accessible that she may gain this perspective.

How might such a history of the teaching of arithmetic be made available? I offer the suggestions.

The National Science Foundation might finance it. There is no better way of improving high school and college mathe-

matics than by improving arithmetic in the elementary school.

The National Council of Teachers of Mathematics or the National Society for the Study of Education could prepare yearbooks around specific aspects.

THE MATHEMATICS TEACHER and THE ARITHMETIC TEACHER could have departments given to history of teaching of arithmetic.

A commercial publisher took a chance in publishing *The World of Mathematics.* More persons could read a History of the Teaching of Arithmetic with profit than can read the more advanced mathematics.

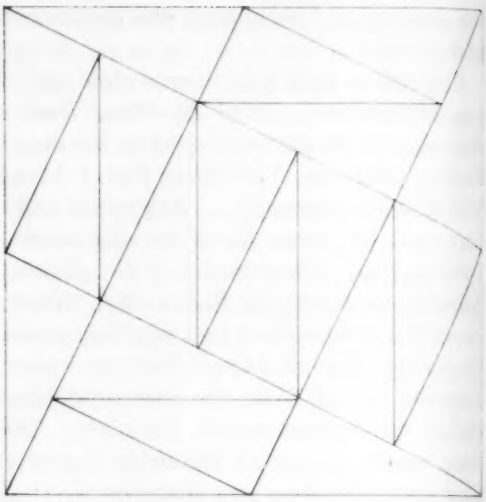
An enthusiast for some new-old method tells us to keep eyes on future—that a momentous break-through in teaching arithmetic is just ahead. Enthusiasts of the past said the same. Historical perspective tells us that the way of improvement is in understanding and marshalling the best of the best, and a steady advance along proven roads. I trust the voice of experience rather than the enthusiast. However, even though a wonder-way were upon us, in comparison with which the past was proved to be a wasteland and void, still I would want a history of the teaching of arithmetic—if only as an obituary to 10,000 years of stumbling.

I use a sentence from the book of Daniel in closing: "Many shall run to and fro, and knowledge shall be increased." In this paper I point out the need for an honest, insightful record of the "to and fro" in arithmetic teaching, and a record of the increase in knowledge.

Key to Per Cent by Color

A very simple geometric pattern used to teach per cent in junior high school math classes is explained below.

I have used it on many occasions for bulletin board work, and the class enjoys working with different color schemes.



First you start with a basic square pattern, 6″×6″.

You connect the mid-points of each side to the opposite corner so as to form two pairs of parallel intersecting lines which will then enclose a smaller inner square.

The unit of measure is a right triangle, four of which are already formed with this construction.

The inside square thus formed has to be subdivided into four (4) more equal right triangles.

In addition, a pair of right triangles is formed on each of the four (4) sides of this inner square.

In all, 20 equal right triangles are formed.

The classes likes to color them so as to have a variation in color schemes, with no two sides containing the same color so as to differentiate them.

The key to the per cent is then left to be discovered by the pupils.

I have found that using from three (3) to six (6) colors to be very effective. Pupils then discover what percentage of the large square is represented by each color. Variations of placement of similarly colored triangles lead to a better understanding.

Contributed by
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Book Reviews

Learning to Use Arithmetic, Grades 1, 2, 3, 4, 5, and 6.

Teachers' Editions, Learning to Use Arithmetic, Grades 1, 2, 3, 4, 5, and 6, Wren, Lynwood F., J. Wayne Wrightstone, Joseph H. Randall, Agnes Gunderson, Joseph J. Urbancek, and George E. Hollister. D. C. Heath and Company, 1958.

The 1958 edition of the *Learning to Use Arithmetic* series offers a carefully developed arithmetic program in an attractive format. The authors' philosophy and the learning experiences advocated throughout the series seem to be consistent with the accepted concepts of a modern arithmetic program.

Teachers' Editions. A most intriguing and unique feature of the teachers' editions for grades 3, 4, 5, and 6 is that each of the editions is a loose-leaf, hard-cover, ring binder. Each of these binders contains a page of the teacher's guide for every page of text. According to the authors, this allows the teacher to organize the guide in whatever way will be most useful to him. A large number of blank spaces are provided on the guide pages. The authors suggest that these blank spaces may be used for additional teaching suggestions and records. The philosophy of flexibility which these teachers' editions may engender will be most welcome to all educators. From a practical point of view it should be noted that these editions are quite large and tend to be relatively heavy in weight. The authors' suggestion that pages could be taken from the binder presupposes that these pages will later be replaced as misplaced and lost pages could make the editions incomplete for future use.

The teachers' editions for grades 1 and 2 are combined with the paper-bound children's editions, with a teachers' guide page appearing opposite each workbook page. Detailed attention is given to building readiness for understanding. Many related activities are suggested, and at spaced intervals throughout the guide, suggested evaluation techniques are indicated.

The wealth of teaching materials and techniques that these guides offer should provide security for the inexperienced teacher and incentives for experimentation for the experienced teacher.

Books 1 and 2. Books 1 and 2 are of a paper-bound, workbook type. Color has been judiciously used and the format is clean and uncluttered. The initial teaching of foundational concepts of the number system is predicated on the basis of child discovery and thoughtful practice. The importance of using concrete materials in the development of each new concept is stressed throughout these books. Emphasis is placed on the integration of experiences with numbers in all phases of the school program.

Books 3, 4, 5, and 6. In Books 3, 4, 5, and 6 the authors have presented each new item in a meaningful, verbal situation based upon an experience which the pupils have had or which they could easily visualize themselves as having. There is a clear development of each new number concept with effective use of visual aids and representative materials. As with most arithmetic texts, the formulation and application of generalizations seem to come too soon after the initiation of each new number concept. Provisions for individual teachers to add material between the initiation and generalization of the new number concept could remedy this difficulty.

The authors have provided a strong problem-solving program. A variety of problems is presented in which children have opportunities to evaluate alternate methods of handling and solving each situation. The problems used throughout the series are characterized by reality and possibility. Many opportunities are provided for developing the skills of estimation and approximation. In fact, it would seem that this series is unique in its early emphasis upon these skills and upon mental arithmetic.

Provision is made for a comprehensive and effective program of maintenance and testing of previously experienced skills and concepts. At the end of each chapter, the authors have included pages devoted to diagnosis, achievement, drill on number facts and skills, and problem solving. These drill procedures and tests would seem to enable the children to evaluate their own progress.

In view of the care with which the authors have worked for meaning and understanding of each new number process, it is somewhat surprising to see the emphasis initially placed upon the process of short division. Formal work in division is introduced in grade 3, but it is not until Chapter 8 of Book 4 that the long form of division is formally presented.

With such comprehensive treatment of and attention to practically all aspects of arithmetic, it is a little disappointing that little or no attention is given to the history and development of our number system. This fascinating story would seem to provide endless opportunities to promote appreciation of the number system and of the breadth and depth of conceptualization.

Books 5 and 6 contain sections entitled "Just for Fun" and "See What You Get." These sections appear at intervals throughout the books as an enrichment feature for all the children. For the most part, however, differentiation of instruction and materials for the slow learners and the more able students is left up to the individual teacher. The texts seem to be designed for the broad middle group of children in each grade. The authors, however, have attempted both in their initial presentation of philosophy and in their suggested teaching procedures to help the individual teacher see his role in the differentiation process.

D. C. Heath's *Learning to Use Arithmetic* should be commended for its interesting and challenging program. The series should go far in helping children discover how wonderful a subject arithmetic can be. With this series the teacher can become a skillful guide, and the child can be the learner using his own brain instead of the teacher's.

ROBERT WELCH (*and Associates*)

Pay the Cashier, Edward W. Dolch. (Second Grade and Up) Champaign, Illinois: The Garrard Press, 1957. Retail, $3.98; School $2.75.

"Pay the Cashier" is a game designed to teach children to make change quickly and accurately. It consists of a deck of cards, a bag of toy coins, some toy paper money—one-dollar, five-dollar, and ten-dollar bills—and a plastic tray made to resemble the money drawer of a cash register. A spinner, designates where—dime store, department store, toy store, supermarket, amusement park, or drugstore—the money is to be spent. The deck of cards is separated according to stores. Players draw from the appropriate pile to learn what to spend at that store. The "customer" either pays the cashier from his own money or hands him a larger amount and gets his change.

The originator of the game suggests that it be used in grades 2 and up. Presumably the gifted students in grade 2 should be able to use it without guidance, but the amounts in some instances are too large for them to understand. Fourth and fifth grades would seem to be the most appropriate placement.

The instructions for playing the game are easily read and understood by fourth and fifth graders. Four children may play at one time. If one player knows how to count change, the others can learn from him, without teacher guidance.

There appears to be one oversight as far as the materials are concerned. The toy coins should more closely resemble real money. The cents are about the size of dimes and the attempt to have them the color of the real cent was unsuccessful. All the coins are too small in relation to the size of real United States coins. Also, contrary to the picture on the package, the game contains no cash register.

The game is interesting to children. It is a welcome addition to the arithmetic table for use by the children in free time. It teaches a skill which is too often neglected in the arithmetic class.

DOROTHY BRUNE